INTRODUCTION TO CONTROLLED TOPOLOGY AND ITS APPLICATIONS

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To Tom Farrell and Lowell Jones

1. Introduction

The idea of using estimates in geometric topology appeared in the early developments of the field. The work that used controlled methods to approach classical problems in geometric topology was done by E. H. Connell and J. Hollingsworth ([30]). In this paper, the authors introduce estimates in computing algebraic obstructions. Later, controlled topology was developed by T. Chapman, S. Ferry and F. Quinn and it became an important tool in the topological classification of topological manifolds. The basic idea in controlled topology is to put an estimate on geometric obstructions. Typical theorems in controlled topology state that if the obstructions are small enough, in an appropriate sense, then they vanish. Another feature of controlled topology is that it captures the homological properties of the obstruction theory. That makes controlled obstructions easier to compute. Essentially, the controlled components reflect the smooth and cellular nature of the obstruction and their complements, the non-controlled components reflect purely topological phenomena. Beyond classification and rigidity theorems controlled topology was applied in the study of group actions on manifolds, equivariant and stratified phenomena.

In the first sections, we present the basic definitions and properties of lower algebraic K-theory. We connect the algebraic definitions with the geometric ones and we present the basic topological applications. The obstructions that appear in topology split into two pieces, the non-controlled and the controlled piece. We continue with the basic constructions of controlled K-theory and its applications. At the same time, we present the non-controlled part of K-theory, namely the Nil-groups. Usually the Nil-groups encode splitting obstructions that correspond to obstructions of reducing the problem to a problem in controlled topology.

There are two applications that will be discussed. The first is the controlled approach to the classical Fundamental Theorem of Algebraic K-theory. The theorem states that

$$Wh(X\times S^1) \cong Wh(X)\oplus\tilde{K}_0(X)\oplus\widetilde{\mathrm{Nil}}(X)\oplus\widetilde{\mathrm{Nil}}(X).$$

Thus, the theorem calculates the K-theory of $X\times S^1$ in terms of the K-theory of X. Using controlled topology, the theorem can be expressed as follows:

$$Wh(X\times S^1) \cong Wh(X\times S^1, p)_c\oplus\widetilde{\mathrm{Nil}}(X)\oplus\widetilde{\mathrm{Nil}}(X)$$

where the $p : X \times S^1 \to S^1$ is the projection map. The group $Wh(X \times S^1, p)_c$ is the controlled Whitehead group and it has the following homological property:

$$Wh(X \times S^1, p)_c \cong Wh(X) \oplus \tilde{K}_0(X)$$

the proof presented here uses geometric methods that were based on the controlled topology of Hilbert cube manifolds ([62]).

The Fundamental Theorem of K-theory calculates the K-theory of spaces that admit a natural map to S^1. Waldhausen's Splitting Theorem calculates the K-theory of spaces that admit a natural map to the unit interval. We interpret Waldhausen's result using controlled topology. Again, the K-theory of such spaces splits into two summands, the one that its the controlled K-theory over the interval and the other that is Waldhausen's Nil-groups.

The second result that is presented here is a controlled approach to Farrell–Jones Isomorphism Conjecture. The conjecture states that certain obstruction groups are isomorphic to homology groups. In other words, the obstruction groups have some homological properties. That means they can be calculated using induction techniques, which in this case are encoded using spectral sequences. More precisely, the conjecture in K-theory predicts that the K-theory of any group can be calculated from the knowledge of the K-theory of its virtually cyclic subgroups (finite subgroups and finite group extensions by the infinite cyclic group) and their lattice structure induced by inclusion and conjugation. The homological part of the Isomorphism Conjecture can be interpreted as a controlled K-group. We show that in the case that the space admits a natural map to the circle or the interval, then the controlled part of the K-theory can be described always as a homology theory with the appropriate control. As an application, we give an alternate proof of the fact that the free groups satisfy the K-theory Isomorphism Conjecture. It should be noted that the existing proofs of the Isomorphism Conjecture use controlled techniques. From the classical proof in [43] to more recent results like in [12], first a theorem in controlled topology is proved and then it is used to prove special cases of the Isomorphism Conjecture.

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2. Very Short Introduction to Algebraic K-Theory

Let R be a ring. To avoid degenerate situations, we assume that R is a ring with identity and that finitely generated free modules have well-defined rank i.e.,

$$R^n \cong R^m \Longleftrightarrow n = m.$$

Remark 2.1. Classical examples of such rings are commutative rings. The proof is very short and elegant: Let R be a commutative ring and M a maximal ideal in R. Then R/M is a field. Assume that $R^n \cong R^m$, as R-modules, which implies $MR^n \cong MR^m$. Then:

$$(R/M)^n = (R/MR)^n \cong R^n/MR^n \cong R^m/MR^m \cong (R/MR)^m = (R/M)^m.$$

But $(R/M)^n$ and $(R/M)^m$ are vector spaces over the field R/M. Since they are isomorphic, they must have the same dimension. Thus $n = m$.

For a group G and a ring R, RG denotes the group ring of G with coefficients in R:

$$RG = \left\{ \sum_{i=1}^{n} r_i g_i : \ r_i \in R, \ g_i \in G, \ \text{for } i = 1, 2, \ldots, n \right\}.$$

For example,

(1) Let $G = \mathbb{Z}^n$ be the free abelian group generated by t_i, $i = 1, \ldots n$. Then $RG = R[t_1^{\pm 1}, \ldots t_n^{\pm 1}]$, the ring of Laurent polynomials on n variables with coefficients in R. In particular, if $G = C_\infty$, the infinite cyclic group generated by t,

$$RC_\infty = R[t, t^{-1}] = \left\{ \sum_{i=-m}^{n} r_i t^i : \ r_i \in R, \ m, n \in \mathbb{N} \right\}.$$

(2) If $G = C_n$, the cyclic group of order n, generated by t then RC_n is a quotient of the polynomial ring on one variable:

$$RC_n \cong R[t]/(t^n - 1).$$

(3) Let $G = F_n$ be the free group generated by x_i, $i = 1, \ldots n$, then RF_n is the Laurent ring on n non-commuting variables with coefficients in R:

$$RF_n = R\{x_1^{\pm 1}, \ldots x_n^{\pm 1}\}.$$

(4) Let $G = H \rtimes_\alpha \mathbb{Z}$ be the semi-direct product of H be $\mathbb{Z}$ with α the action of the generator of $\mathbb{Z}$ on H. Then $RG \cong RH_\alpha[t, t^{-1}]$, the twisted Laurent polynomial ring of RH. More precisely, for $r, s \in RH$,

$$(rt^n)(st^m) = r\alpha^{-n}(s)t^{n+m}.$$

Remark 2.2. If R has well-defined rank for finitely generated free modules, so does RG, for any group G. The proof is the same as in Remark 2.1 where the maximal ideal M is replaced by the augmentation ideal I:

$$I = \mathrm{Ker}(\varepsilon : RG \to R)$$

where ε is the augmentation homomorphism.

2.1. K_0, K_1 **and Nil-groups of rings.** By a module over a ring R we will mean a **left module**. Everything works the same when right modules are chosen instead. The basic definitions are included in [6] and [78].

Definition 2.3. The group $K_0(R)$ is defined as $\mathcal{F}_R/N_R$ where:

- $\mathcal{F}_R$ is the free abelian group generated by the isomorphism classes of finitely generated projective R-modules.
- $\mathcal{N}_R$ is the subgroup of $\mathcal{F}_R$ generated by elements of the form $[P_0 \oplus P_1] - [P_0] - [P_1]$.

Examples. Some very basic examples:

(1) If F is a field, then $K_0(F) \cong \mathbb{Z}$, and the isomorphism is given by the dimension.
(2) For $G = \mathbb{Z}^n$, $K_0(\mathbb{Z}G) \cong \mathbb{Z}$ (Bass–Heller–Swan, [7]).
(3) If $G = F_n$, $K_0(\mathbb{Z}F_n) \cong \mathbb{Z}$ (Gersten, [51]).
(4) If R is the ring of integers in a number field, then $K_0(R)$ is isomorphic to the class group of R ([78]).
(5) Let X be a compact Hausdorff topological space and $C(X)$ be the ring of continuous functions from X to $\mathbb{R}$. Then $K_0(C(X)) \cong K^0(X)$ where $K^0(X)$ is Atiyah's geometric K-group of real vectors bundles over X (Swan, [86], [78]).

For the definition of $K_1(R)$, let $GL(n, R)$ be the group of invertible $n \times n$-matrices with entries in R. Let $E(n, R)$ be the subgroup generated by elementary matrices i.e., matrices that are formed from the identity by adding the multiple of a row (or column) to another row (or column). More precisely, $E(n, R)$ is generated by matrices $e_{i,j}(r)$, with $1 \leq i, j \leq n$, $i \neq j$, $r \in R$, of the form:

$$e_{i,j}(r) = \begin{pmatrix} 1 & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & 1 & \dots & r & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \dots & \dots & \dots & 1 \end{pmatrix}$$

where r is the (i, j)-position. There are monomorphisms:

$$GL(n, R) \to GL(n + 1, R), \quad E(n, R) \to E(n + 1, R)$$

induced by the map

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Taking direct limits:

$$GL(R) = \varinjlim GL(n,R) = \bigcup_{n=1}^{\infty} GL(n,R), \quad E(R) = \varinjlim E(n,R) = \bigcup_{n=1}^{\infty} E(n,R).$$

Lemma 2.4 (Whitehead's Lemma)**.** *$[GL(R), GL(R)] = E(R)$.*

Proof. The following holds in $GL(2n, R)$ for each invertible $n\times n$-matrix u:

$$\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ u^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & 1-u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1-u^{-1} \\ 0 & 1 \end{pmatrix}$$

So that the matrix on the left hand side of the equation is elementary. For the commutators, we see that

$$[g,h] = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}\begin{pmatrix} (hg)^{-1} & 0 \\ 0 & hg \end{pmatrix}$$

which implies that the commutators belong to $E(R)$. □

Definition 2.5. For a ring R define $K_1(R) = GL(R)/E(R)$.

Remark 2.6.

(1) Whitehead's Lemma implies that $K_1(R)$ is abelian.
(2) $K_1(R)$ can be thought as the obstruction for solving a system of linear equations with coefficients in R using Gauss operations of the third type (adding a multiple of an equation to another equation).

Examples

(1) If F is a field, then the determinant map induces and isomorphism $K_1(F) \cong F \setminus \{0\}$.
(2) In general, if R is a commutative ring then the determinant induces a homomorphism:

$$\det : K_1(R) \to U(R),$$

where $U(R)$ are the units of R. The map det is a split surjection (Proposition 2.2.1 [78]). We write $SK_1(R) = \ker(\det)$. When $R = \mathbb{Z}$, the determinant map induces an isomorphism $K_1(\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.
(3) If R is the ring of integers in a number field, $SK_1(R)$ vanishes and $K_1(R)$ is the group of units of R ([78]).
(4) For a compact Hausdorff topological space X and $C(X)$ be the ring of continuous functions from X to $\mathbb{R}$, Then $K_1(C(X)) \cong K^1(X)$ Atiyah's geometric K-group of real vectors bundles over X ([78]). Actually, the Serre–Swan's Theorem states that the category of real vector bundles over X is equivalent to the category of finitely generated projective modules over $C(X)$.

Let $\alpha : R \to R$ be a ring automorphism. Let M be an R-module. A map $f : M \to M$ is called α-linear if:

- $f(m+n) = f(m) + f(n)$, for all m, n in M.
- $f(rm) = \alpha(r)f(m)$, for all $r \in R$, $m \in M$.

Consider pairs (P, ν) where P is a finitely generated projective R-module and ν is an α-linear nilpotent endomorphism of P. We call such pairs *nil pairs.* A morphism of nil pairs

$$f : (P, \nu) \to (Q, \mu)$$

is an R-linear homomorphism $f : P \to Q$ making the diagram commutative:

$$\begin{array}{ccc} P & \xrightarrow{f} & Q \\ \nu \downarrow & & \downarrow \mu \\ P & \xrightarrow{f} & Q \end{array}$$

Exact sequences of nil pairs are given by:

$$0 \to (P_0, \nu_0) \xrightarrow{f} (P, \nu) \xrightarrow{g} (P_1, \nu_1) \to 0 \quad (E)$$

where the following diagram of exact sequences commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_0 & \xrightarrow{f} & P & \xrightarrow{g} & P_1 & \longrightarrow & 0 \\ & & \nu_0 \downarrow & & \nu \downarrow & & \nu_1 \downarrow & & \\ 0 & \longrightarrow & P_0 & \xrightarrow{f} & P & \xrightarrow{g} & P_1 & \longrightarrow & 0 \end{array}$$

Notice that, in general, the exact sequence (E) does not split i.e., (P, ν) is not the direct sum of (P_0, ν_0) and (P_1, ν_1). Actually, if we identify $P \cong P_0 \oplus P_1$ then:

$$\nu = \begin{pmatrix} \nu_0 & \chi \\ 0 & \nu_1 \end{pmatrix} : P_0 \oplus P_1 \to P_0 \oplus P_1,$$

where $\chi : P_1 \to P_0$ is any α-linear homomorphism.

Definition 2.7. The group $\mathrm{Nil}(R, \alpha)$ is defined as $\mathcal{F}N_R / \mathcal{N}N_R$ where

- $\mathcal{F}N_R$ is the free abelian group generated by the isomorphism classes of nil objects (P, ν).
- $\mathcal{N}N_R$ is the subgroup generated by:
  - Elements of the form
    $$[P, \nu] - [P_0, \nu_0] - [P_1, \nu_1]$$
    for each exact sequence (E).
  - Elements $[F, 0]$ where F is a finitely generated free R-module.

The reduced Nil-group $\widetilde{\mathrm{Nil}}(R, \alpha)$ is defined to be the subgroup of $\mathrm{Nil}(R, \alpha)$ generated by elements of the form $[F, \nu]$ where F is a finitely generated free R-module. We write $\widetilde{\mathrm{Nil}}(R)$ for $\widetilde{\mathrm{Nil}}(R, \mathrm{id}_R)$.

Remark 2.8.

(1) Suppose that α has finite order. Then if $\widetilde{\mathrm{Nil}}(R,\alpha) \neq 0$ it is infinitely generated (Farrell for $\alpha = 1$ ([39]), Ramos ([76]) and Grunewald ([54], [55]) for the general case). The question of finiteness or not of Nil-groups is open when α has infinite order.

(2) Remember that a ring R is called *regular coherent* if every finitely presented R-module M admits a resolution by finitely generated projective modules. The ring is called *regular Noetherian* is if in addition any finitely generated R-module is finitely presented. If R is a regular coherent ring then $\widetilde{\mathrm{Nil}}(R,\alpha)$ vanishes (Bass ([6], [7]) and Waldhausen ([90])).

(3) Group rings of finitely generated free groups are regular coherent. Group rings of finitely generated free abelian groups are regular Noetherian. Even though twisted polynomial rings of Noetherian rings are Noetherian, the same is not true for coherent rings ([52]).

The remarks leave open two conjectures.

Conjecture 1. *Let R be any ring and $\alpha : R \to R$ a ring automorphism of infinite order. Then $\widetilde{Nil}(R,\alpha)$ is either zero or infinitely generated.*

Conjecture 2. *Let R be a ring of finite cohomological dimension (i.e., there is a number n so that every module has a resolution of finite length at most n). Then $\widetilde{Nil}(R,\alpha) = 0$ for each ring automorphism α.*

For the second conjecture, recent developments are presented in [14] and [15].

Remark 2.9. The following question has been asked by Tom Farrell: Given a ring R and a ring automorphism α of R, is there a ring S so that $\widetilde{\mathrm{Nil}}(R,\alpha) \cong \widetilde{\mathrm{Nil}}(S)$? For partial results the reader should check [5]. Related to the above question is the following: Let R be a ring so that $\widetilde{\mathrm{Nil}}(R) = 0$. Is it true that $\widetilde{\mathrm{Nil}}(R,\alpha) = 0$ for any ring automorphism α of R?

Remark 2.10. There is a common pattern in the definition of the K-groups given above. The K_0-groups should be considered as the Grothedieck groups of matrices that are projections, the K_1-groups of invertible matrices and the Nil-groups of nilpotent matrices.

2.2. **Algebraic K-theory in Topology.** We present some of the basic applications of algebraic K-theory to topology. For the topological applications we need certain quotients of the algebraic K-groups.

Definition 2.11. Let R be a ring with 1. Let $\mathbb{Z} \to R$ be the ring homomorphism induced by mapping 1 to the identity. Define the reduced K_0-group

$$\tilde{K}_0(R) = \mathrm{Coker}(K_0(\mathbb{Z}) \to K_0(R)).$$

Actually,

$$\tilde{K}_0(R) = K_0(R)/\langle[R]\rangle$$

where $\langle[R]\rangle$ is the infinite cyclic subgroup generated by the free R-module $[R]$ of rank 1.

Remark 2.12.

(1) An element $[P]$ in $\tilde{K}_0(R)$ is zero iff P is finitely generated *stably free module* i.e., there is a finitely generated free module F such that $P \oplus F$ is free.
(2) For any module P there is Q such that $P \oplus Q$ which is free. Then there is an isomorphism

$$P \oplus (Q \oplus P \oplus Q \oplus \dots) \cong P \oplus Q \oplus P \oplus Q \oplus \dots \Longrightarrow P \oplus F \cong F,$$

where F is a countably generated free module. Thus if the finite generation condition is omitted in (1), every projective module is stably free (Eilenberg swindle).

Definition 2.13. A space Y is called finitely dominated space iff there is a finite complex K and maps

$$Y \xrightarrow{d} K \xrightarrow{u} Y,$$

such that $u \circ d \simeq \mathrm{id}_Y$.

Remark 2.14. Let Y be a finitely dominated space as above. Then the map

$$e : K \xrightarrow{u} Y \xrightarrow{d} K,$$

satisfies

$$e^2 = dudu = d(ud)u \simeq d(\mathrm{id}_Y)u = du = e.$$

Thus e is an idempotent up to homotopy (for the general theory of homotopy idempotents see [37], [57]).

Let Y be a finitely dominated connected with $\pi_1(Y) = \pi$ space and $\tilde{Y}$ its universal cover. By adding a finite number of cells we arrange that the finite complex K that dominates Y has $\pi_1(K) = \pi$ ([91]). Then $C_*(\tilde{Y})$ is a $\mathbb{Z}\pi$-chain complex which is dominated by a finite complex of free modules namely $C_*(\tilde{K})$. That means there are chain maps

$$C_*(\tilde{Y}) \xrightarrow{u} C_*(\tilde{K}) \xrightarrow{d} C_*(\tilde{Y})$$

and a chain homotopy $d \circ u \simeq \mathrm{id}_{C_*(\tilde{Y})}$. Then $C_*(\tilde{Y})$ is chain homotopy equivalent to a chain complex P_* of finitely generated projective modules (Wall ([91], [92])):

$$P_* : \ \dots 0 \to P_n \to P_{n-1} \to \dots P_1 \to P_0.$$

Thus a finitely dominated space determines an element

$$\sigma(Y) = \sum_{i=0}^{n} (-1)^i [P_i] \in \tilde{K}_0(\mathbb{Z}\pi).$$

Theorem 2.15 (Wall's Finiteness Obstruction)**.** *A finitely dominated space Y determines an element in $\sigma(Y) \in \tilde{K}_0(\mathbb{Z}\pi)$ which vanishes iff Y is homotopy equivalent to a finite complex.*

Remark 2.16. Using Wall's finiteness obstruction, we can interpret finitely dominated spaces as projective modules, up to homotopy, (they determine homotopy idempotents an in Remark 2.14) and the finite complexes as free modules.

A more geometric application of the finiteness obstruction comes from Siebenmann's Thesis ([79]). The question that this work addresses is when an end in a manifold can be completed i.e., it can be realized as the interior of a compact manifold with boundary. More precisely, the compact manifold with boundary $\bar{M}$ is a completion of a manifold M if $M = \bar{M} - \partial\bar{M}$.

Definition 2.17. An end ε in a topological space X is called tame if:

(1) The end ε is stable: There is a sequence of connected open neighborhoods of ε such that there is a cofinal subsequence of the fundamental groups

$$G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} \dots G_n \xleftarrow{f_n} \dots$$

which induces a sequence of isomorphism:

$$\mathrm{Im}(f_1) \xleftarrow{\cong} \mathrm{Im}(f_2) \xleftarrow{\cong} \dots \xleftarrow{\cong} \mathrm{Im}(f_n) \xleftarrow{\cong} \dots$$

(2) There are arbitrarily small neighborhoods of ε that are finitely dominated.

For a complete exposition of the algebraic theory of ends a reference is Hughes–Ranicki ([63]).

Theorem 2.18 (Siebenmann's Thesis)**.** *Let M^n be a manifold with $n \geq 6$ and ε a tame end of M. Then ε can be completed iff an obstruction $\sigma(\varepsilon) \in \tilde{K}_0(\mathbb{Z}\pi_1(\varepsilon))$ vanishes, where $\pi_1(\varepsilon)$ is the fundamental group of the end.*

Now we present some geometric applications of K_1. Again, for geometric applications we need a quotient of the algebraic K_1.

Definition 2.19. Let G be a group. The Whitehead group of G is defined as

$$\mathrm{Wh}(G) = K_1(\mathbb{Z}G)/\{\pm g : \ g \in G\}.$$

Here $\pm g \in GL(1, \mathbb{Z}G)$, for all $g \in G$.

The geometric applications of $\mathrm{Wh}(G)$ are derived from the fact that a strong deformation retraction:

$$f : K \to L$$

of connected finite CW-complexes determines an element in $\mathrm{Wh}(\pi_1(L))$ as follows ([29], [77]): Let $\pi = \pi_1(L)$. Choose a CW-decomposition of K and L so that L is a subcomplex of K. Then the relative chain complex $C_* = C_*(\tilde{K}, \tilde{L})$ is a $\mathbb{Z}\pi$-chain complex, where $\tilde{K}$ (resp. $\tilde{L}$) is the universal cover of K (resp. L). Liftings of the cells to the universal covers determines a $\mathbb{Z}\pi$-basis for C_*. The ambiguity of the choice of liftings is given by elements $\pm g$, $g \in \pi$ (the group element for the particular choice of the lift and the sign for the choice of the orientation of the cell). That is why we consider the quotient of $K_1(\mathbb{Z}\pi)$. Then C_* is a based, finite, contractile chain complex. Thus there is an isomorphism ([77]):

$$\delta + \partial : C_{\text{even}} \to C_{\text{odd}}$$

where ∂ is the boundary map, δ is the chain contraction and C_{even} (resp. C_{odd}) is the direct sum of the even (resp. odd) part of C_*. Using the bases determined by the cells, $\delta + \partial$ determines an invertible matrix with entries in $\mathbb{Z}\pi$. We define the *torsion* of f to be that matrix: $\tau(f) = [\delta + \partial] \in \mathrm{Wh}(\pi)$. In general, if $f : K \to L$ is a homotopy equivalence then $\tau(f) = f_*(\tau(M(f), K))$, where $M(f)$ is the mapping cylinder of f.

Definition 2.20. Let L be a space. An elementary expansion K of L corresponds to the attaching of a ball to L along a face of the ball. More precisely, K is formed as the push-out

$$\begin{array}{ccc} S^{n-1} & \xrightarrow{q} & L \\ \downarrow & & \downarrow \\ D^n & \xrightarrow{Q} & K \end{array}$$

In this case, we say that L is formed by an elementary collapse of K. A simple homotopy equivalence $f : K \to L$ is a homotopy equivalence that can be described as a finite sequence of elementary expansions and collapses.

Remark 2.21. Simple homotopy equivalences were introduced by J. H. C. Whitehead ([93], [94]). His goal was to study homotopy equivalences combinatorially.

Theorem 2.22. *A homotopy equivalence $f : K \to L$ between finite CW-complexes is simple iff $\tau(f) = 0 \in Wh(\pi_1(L))$.*

The importance of the Whitehead torsion comes with its applications to the topology of manifolds.

Definition 2.23. A cobordism $(W; M, M')$ is a manifold W such that $\partial W = M \coprod M'$. An h-cobordism is a cobordism such that the inclusion maps

$$M \hookrightarrow W, \quad M' \hookrightarrow W$$

are homotopy equivalences.

The following works in all categories of manifolds (Diff, PL or Top).

Theorem 2.24 (s-Cobordism Theorem)**.** *Let $dim(M) \geq 5$. Then there is a bijection:*

$$\{h - \textit{cobordisms over } M\} \longrightarrow Wh(\pi_1(M)), \quad (W; M, M') \mapsto \tau(W, M).$$

In particular $\tau(W, M) = 0 \in Wh(\pi_1(M))$ iff $W \cong M \times [0, 1]$ which implies that $M \cong M'$.

Remark 2.25.

(1) The equivalences in the s-cobordism theorem are equivalences in the given category (diffeomorphisms, PL-homeomorphisms, homeomorphisms respectively).
(2) The h-cobordism theorem (which is the s-cobordism theorem when M is simply connected) was the main ingredient in the proof of the Poincaré Conjecture for $n \geq 5$ by Smale ([81]).

For the calculations of the reduced K_0 and the Whitehead group, there is the following conjecture.

Conjecture 3. *Let G be a torsion free group. Then*

$$\tilde{K}_0(\mathbb{Z}G) = Wh(G) = 0.$$

Algebraically, Conjecture 3 is related to the following conjecture.

Conjecture 4. *Let G be a torsion free group. Then*

$$(\mathbb{Z}G)^\times = \pm G$$

i.e., the units of $\mathbb{Z}G$ are exactly the elements of G and their negatives. Also, $\mathbb{Z}G$ contains no non-trivial nilpotent elements.

Remark 2.26. Notice that if $x \in \mathbb{Z}G$ is nilpotent then $1 - x$ is a unit.

The geometric definition of Nil-groups is given in [62] and [72]. Let X be a finite CW-complex. A *nil-pair* over X, (Y, f) consists of a finite CW-complex Y containing X and $f : Y \to Y$ such that:

(1) $f_X = \mathrm{id}_X$.
(2) There is $s > 0$ such that $f^s \simeq r$ rel$X$, where r is a retraction to X (homotopy nilpotent condition).

In [72] it was shown that, if (1) holds, the second condition is equivalent to

(2*) The inclusion map $X \times S^1 \hookrightarrow T(f)$ is a homotopy equivalence.

Define $(Y, f) \sim (Y' f')$ if there is a homotopy equivalence $f : Y \to Y'$, rel$X$ such that $Ff \simeq f'$ rel$X$. Let $\mathcal{F}(X)$ be the free group on the equivalence classes of nil-pairs over X. Let $\mathcal{N}(X)$ be the subgroup generated by:

(1) For two nil-pairs (Y_i, f_i), $i = 1, 2$,

$$[Y_1 \cup_X Y_2, f_1 \cup f_2] - [Y_1, f_1] - [Y_2, f_2]$$

(2) The nil-pairs $[Y, r]$, with $r : Y \to X$ a retraction.

Then $\widetilde{\mathrm{Nil}}(X) = \mathcal{F}(X)/\mathcal{N}(X)$.

The isomorphism between the algebraic and geometric Nil-groups is given:

$$\widetilde{\mathrm{Nil}}(X) \to \widetilde{\mathrm{Nil}}(\pi_1(X)), \;\; (Y, f) \mapsto (C_*(\bar{Y}, \tilde{X}), f_*)$$

where $\tilde{X}$ is the universal cover of X, $\bar{Y}$ is the pull-back of the universal cover of X under the retraction $r : Y \to X$, $C_*(\bar{Y}, \tilde{X})$ is a finite $\mathbb{Z}\pi_1(X)$-complex of free modules. Notice that f_* is a chain homotopy nilpotent map. For the precise way that a pair $(C_*(\bar{Y}, \tilde{X}), f_*)$ determines an element of $\widetilde{\mathrm{Nil}}(\mathbb{Z}\pi_1(X))$ see [77].

Nil-groups appear as obstructions to splitting problems. We will present a brief summary of the applications after introducing the Fundamental Theorem of Algebraic K-theory.

2.3. **The Fundamental Theorem of Algebraic K-theory.** For a ring R, the Fundamental Theorem of Algebraic K-theory computes the K-theory of $R[t, t^{-1}]$ in terms of the K-theory of R.

Theorem 2.27 (Fundamental Theorem of Algebraic K-theory)**.** *For a ring R*

$$K_1(R[t, t^{-1}]) \cong K_1(R) \oplus K_0(R) \oplus \widetilde{Nil}(R) \oplus \widetilde{Nil}(R).$$

For the geometrically significant groups,

$$Wh(G \times \mathbb{Z}) \cong Wh(G) \oplus \tilde{K}_0(\mathbb{Z}G) \oplus \widetilde{Nil}(\mathbb{Z}G) \oplus \widetilde{Nil}(\mathbb{Z}G).$$

Remark 2.28.

(1) There is an analogue of the Fundamental Theorem for higher K-theory ([65], [82]).

(2) For twisted Laurent rings the Fundamental Theorem of Algebraic K-theory has the form ([42]):

$$K_1(R) \xrightarrow{1-\alpha_*} K_1(R) \to K_1(R_\alpha[t, t^{-1}]) / \left(\widetilde{\mathrm{Nil}}(R, \alpha) \oplus \widetilde{\mathrm{Nil}}(R, \alpha^{-1})\right) \to K_0(R) \xrightarrow{1-\alpha_*} K_0(R)$$

(3) The Fundamental Theorem of Algebraic K-theory measures the failure of K-theory to be a homology theory. In particular, the Nil-terms are exactly the measure of that failure.

Let X be a finite CW-complex (or a compact ANR ([25])) with $\pi_1(X) = G$. Then there is a transfer map

$$tr^n : Wh(G \times \mathbb{Z}) \to Wh(G \times \mathbb{Z})$$

defined geometrically as follows: Let $\tau = (f : Y \to X)$ be represented by an strong deformation retraction. Form the pull-back diagram

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\bar{f}} & X \times S^1 \\ \big\downarrow & & \big\downarrow \mathrm{id}_X \times n \\ Y & \xrightarrow{f} & X \times S^1 \end{array}$$

where n is the map that sends an element of S^1 (considered as the unit circle in the complex plane) to its n-th power. Define $tr^n(\tau) = \tau(\bar{f})$. The summands of the splitting in the fundamental theorem behave as follows ([72], [77]):

(1) If $x \in Wh(G)$, then $tr^n(x) = nx$.

(2) If $x \in \tilde{K}_0(\mathbb{Z}G)$, $tr^n(x) = x$. Thus the elements of $\tilde{K}_0(\mathbb{Z}G)$ are exactly the elements of $Wh(G)$ that are transfer invariant.

(3) If $x \in \widetilde{\mathrm{Nil}}(\mathbb{Z}G) \oplus \widetilde{\mathrm{Nil}}(\mathbb{Z}G)$, then there is an $s > 0$ such that $tr^n(x) = 0$ for all $n > s$.

The Fundamental Theorem can be used for extending the definition of K_i-groups with $i < 0$. More precisely, inductively, $K_i(R)$ is defined to be the subgroup of the transfer invariant elements of $K_{i+1}(R)$ for $i \leq 1$. The Fundamental Theorem of Algebraic K-theory extends to the negative K-groups. The formal details for the delooping of the algebraic K-theory spectrum are given in [71].

2.4. **Waldhausen's Splitting Theorem.** Let A_i, $i = 0, 1$ be two rings containing R as a subring. Assume that $A_i = R \oplus B_i$ as R-bimodules. Let Λ be the push out in the category of R-algebras:

$$\begin{array}{ccc} R & \longrightarrow & A_1 \\ \downarrow & & \downarrow \\ A_2 & \longrightarrow & \Lambda \end{array}$$

As an R-bimodule Λ splits as:

$$\Lambda = R \oplus B_1 \oplus B_2 \oplus B_1 \otimes B_2 \oplus B_2 \otimes B_1 \oplus \dots$$

where $\otimes$ means $\otimes_R$. The multiplication is given by concatenation and using the ring structure of A_i.

Theorem 2.29 (Waldhausen ([90]))**.** *With the above notation, there is an exact sequence:*

$$\dots \to K_1(R) \to K_1(A_1) \oplus K_1(A_2) \to K_1(\Lambda)/N_1 \to K_0(R) \to \dots$$

where $N_1 = \widetilde{Nil}_0(R; B_0, B_1)$ *is Waldhausen's Nil-group.*

Remark 2.30. For more general splitting theorems, see [83] and [84].

We will give the definition of Waldhausen's Nil-groups. For the triple $(R; B_0, B_1)$ with R a ring and B_i, $i = 0, 1$, two R-bimodules, consider quadruples (P, Q, p, q) where P and Q are finitely generated projective R-modules and

$$p : P \to Q \otimes B_0, \quad q : Q \to P \otimes B_1$$

(here $\otimes$ means $\otimes_R$) are two R-maps such that the compositions

$$\begin{array}{ccccccc} P & \xrightarrow{p} & Q \otimes B_0 & \xrightarrow{q \otimes \mathrm{id}} & P \otimes B_1 \otimes B_0 & \dots \\ Q & \xrightarrow{q} & P \otimes B_1 & \xrightarrow{p \otimes \mathrm{id}} & P \otimes B_0 \otimes B_1 & \dots \end{array}$$

are eventually 0. Exact sequences have the obvious meaning. As before, define $\mathrm{Nil}_0(R; B_0, B_1)$ to be the free abelian group generated by isomorphism classes of quadruples as above modulo the additive relation resulting from the exact sequences. Notice that there are homomorphisms:

$$\begin{array}{ll} K_0(R) \times K_0(R) \to \mathrm{Nil}_0(R; B_0, B_1), & (P, Q) \mapsto (P, Q, 0, 0) \\ \mathrm{Nil}_0(R; B_0, B_1) \to K_0(R) \times K_0(R), & (P, Q, p, q) \mapsto (P, Q) \end{array}$$

which make $K_0(R)\times K_0(R)$ a summand of $\mathrm{Nil}_0(R; B_0, B_1)$. Define

$$\widetilde{\mathrm{Nil}}_0(R; B_0, B_1) = \mathrm{Nil}_0(R; B_0, B_1)/K_0(R)\times K_0(R).$$

Remark 2.31.

(1) The quadruples defined above form an exact category $\mathcal{N}il(R; B_0, B_1)$. In general, define

$$\widetilde{\mathrm{Nil}}_i(R; B_0, B_1) = K_i(\mathcal{N}il(R; B_0, B_1))/K_i(R)\times K_i(R).$$

The splitting theorem states that $\widetilde{\mathrm{Nil}}_{i-1}(R; B_0, B_1)$ is a summand of $K_i(\Lambda)$. Thus there is a dimension shift. Waldhausen's splitting theorem holds for all $i \in \mathbb{Z}$.

(2) For more information on the structure of lower Waldhausen Nil-groups see [66], [67].

3. Controlled Topology

Let B be a metric space and $p : X \to B$ be a continuous map and $\varepsilon > 0$. A homotopy

$$F : Y\times[0, 1] \to X$$

is called a $p^{-1}(\varepsilon)$ homotopy if

$$\sup\{\mathrm{diam}(pF(\{y\}\times[0, 1])) : y \in Y\} < \varepsilon.$$

That means that the paths determined by the homotopy have diameter less that ε when projected to B. In this case we write $F_0 \simeq_{p^{-1}(\varepsilon)} F_1$.

Remark 3.1. More generally, the assumption that B is metric space is not needed. For a general topological space B and an open coven $\mathcal{U}$ of B, the homotopy F is called $p^{-1}(\mathcal{U})$-homotopy if, for each $y \in Y$, there is $U_y \in \mathcal{U}$ such that

$$pF(\{y\}\times[0, 1]) \subset U_y.$$

Definition 3.2. Let $p : X \to B$ be as above and $\varepsilon > 0$. A map $f : Y \to X$ is called a $p^{-1}(\varepsilon)$-homotopy equivalence if there is $g : X \to Y$ such that

$$g\circ f \simeq_{(pf)^{-1}(\varepsilon)} \mathrm{id}_Y, \quad f\circ g \simeq_{p^{-1}(\varepsilon)} \mathrm{id}_X.$$

A continuous map $f : Y \to X$ is called a controlled homotopy equivalence (with control in B) if for each $\delta > 0$, f is a $p^{-1}(\delta)$-homotopy equivalence.

Definition 3.3. A proper map $f : Y \to X$ is called cell-like (CE-map) if for each $x \in X$ and each neighborhood U of $f^{-1}(x)$, the inclusion map $f^{-1}(x) \hookrightarrow U$ is null-homotopic.

Remark 3.4.

(1) If $f^{-1}(x)$ is an ANR for each $x \in X$, then a proper map f is a CE-map iff $f^{-1}(x)$ is contractible for all $x \in X$. In particular, a proper cellular map f between CW-complexes is a CE-map iff $f^{-1}(x)$ is contractible for all $x \in X$.

(2) Cell-like maps are the homotopy analogues of homeomorphisms. In a homeomorphism we require that the inverse image of a point is a point. In a CE-map we require that the inverse image of a point has the homotopy type of a point.
(3) If $f : Y \to X$ is a CE-map between locally compact ANR's then f is a controlled homotopy equivalence with control in X.
(4) The basic properties of CE-maps are presented in [68].

Controlled homotopy equivalences and CE-maps are closely related to the controlled theory of fibrations.

Definition 3.5. Let $p : X \to B$ be map to a metric space B and $\varepsilon > 0$. A map $q : E \to X$ is called a $p^{-1}(\varepsilon)$-fibration if the lifting problem

$$\begin{array}{ccc} Z\times\{0\} & \xrightarrow{f} & E \\ {\scriptstyle i}\downarrow & \nearrow{\scriptstyle \hat{F}} & \downarrow{\scriptstyle q} \\ Z\times[0,1] & \xrightarrow{F} & X \end{array}$$

has a solution $\hat{F}$ such that:

- The top triangle commutes: $\hat{F}\circ i = f$.
- The bottom diagram $p^{-1}(\varepsilon)$-commutes: $d(pq\hat{F}(z,t), pF(z,t)) < \varepsilon$, for all $(z,t) \in Z\times[0,1]$.

The map q is called an approximate fibration if it is a δ-fibration for each δ.

The properties of approximate fibrations are proved in [32] and [33]. The following result follows from the definitions ([68]).

Theorem 3.6. *A map $f : Y \to X$ between locally compact ANR's is a CE-map iff it is a homotopy equivalence and an approximate fibration.*

The following constructions are from [20]. Let X be a compact ANR and $p : X \to B$ be a control map to metric space B and $\varepsilon > 0$. Two $p^{-1}(\varepsilon)$-strong deformation retractions

$$f_i : Y_i \to X, \quad i = 1, 2,$$

with Y_i compact ANR's, are called equivalent, denoted $f_1 \sim f_2$, if there is a compact ANR Z such that the diagram:

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & Y_1 \\ {\scriptstyle \beta}\downarrow & & \downarrow{\scriptstyle f_1} \\ Y_2 & \xrightarrow{f_2} & X \end{array}$$

commutes up to $p^{-1}(\varepsilon)$-homotopy rel$X$. The above relation generates an equivalence relation on the set of $p^{-1}(\varepsilon)$-strong deformation retractions to X.

The ε-controlled Whitehead group of X, $Wh(X,p)_\varepsilon$ is defined as the group with elements equivalence classes of $p^{-1}(\varepsilon)$-sdr's whose inverses in the classical Whitehead group are $p^{-1}(\varepsilon)$-sdr's. The group operation is given by push-outs:

$$(f_1 : Y_1 \to X) + (f_2 : Y_2 \to X) = (f_1 \cup f_2 : Y_1 \cup_X Y_2 \to X).$$

The *controlled Whitehead group* is defined as the inverse limit

$$Wh(X,p)_c = \varprojlim Wh(X,p)_\varepsilon.$$

Theorem 3.7 ([19]). *Let $p : X \to B$ be an approximate fibration with X a compact ANR and B a locally compact ANR. Then a strong deformation retraction $f : Y \to B$ represents an element in $Wh(X,p)_c$ iff $p \circ f$ is an approximate fibration.*

Remark 3.8.

(1) There is a more algebraic approach to controlled topology given in [74] and [75]. That approach is useful for complete calculations of the controlled groups.
(2) There are analogues of the h-cobordism theorem and the finite domination theorem in the controlled setting ([19], [74], [75]).

The controlled groups capture the homological properties of K-theory. In special cases, the homological properties of the controlled groups imply the vanishing of certain obstructions.

Theorem 3.9 (The sucking principle or squeezing ([22], [24], [59])). *Let X be a compact ANR. For each $\varepsilon > 0$, there is $\delta > 0$ such that every ε-fibration, with control in X, $p : E \to X$, with E a compact ANR, is δ-homotopic to an approximate fibration.*

Theorem 3.10 (The thin h-cobordism theorem ([19], [74], [75]). *Let M^n be a compact manifold with $n \geq 5$. Then for each $\varepsilon > 0$ there is $\delta > 0$ such that every ε-h-cobordism $(W; M, M')$ (i.e., such that the strong deformations retractions $W \to M$ and $W \to M'$ are ε-homotopy equivalences with control in M) is a δ-product (i.e., there is a homeomorphism $h : W \to M \times [0,1]$ such that $p \circ h$ is δ-close to the retraction of W to M, here $p : M \times [0,1] \to M$ is the projection map).*

Theorem 3.11 (The α-approximation theorem ([22], [28], [48])). *Let M^n be a compact manifold with $n \geq 5$. Then for each $\varepsilon > 0$ there is $\delta > 0$ such that every ε-homotopy equivalence $f : N^n \to M^n$, with control in M^n, and N^n a compact manifold is δ-homotopic to a homeomorphism.*

There are versions of the approximation theorems where the control map is not the identity ([41], [75]). In these cases, the map is required to be at least a stratified fibration and that the problem can be solved in the fibers.

3.1. **The Fundamental Theorem of Algebraic K-theory Revisited.** Using controlled topology, the Fundamental Theorem can be written as follows ([62]):

Theorem 3.12. *For a compact ANR X, there is a splitting:*

$$Wh(X\times S^1) \cong Wh(X\times S^1, p)_c \oplus \widetilde{Nil}(X) \oplus \widetilde{Nil}(X),$$

where $p : X\times S^1 \to S^1$ is the projection map. Furthermore ([64]),

$$Wh(X\times S^1, p)_c \cong Wh(X) \oplus \tilde{K}_0(X).$$

We outline the geometric proof of the Fundamental Theorem of algebraic K-theory. We will use Hilbert cube manifolds ([26]). Remember that the Hilbert cube is defined as

$$Q = \prod_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right],$$

with the induced metric. Hilbert cube manifolds are separable metric spaces that have the local structure of Q. Some basic facts on Q-manifolds ([26]):

(1) If X is a locally compact ANR, then $X\times Q$ is a Q-manifold.
(2) Any map $f : X \to Y$ with X a locally compact ANR and Y a locally compact Q-manifold, can be approximated by an embedding.
(3) Any homotopy $h : X\times I \to Y$, with X and Y as above, can be approximated by an isotopy.
(4) The α-approximation theorem holds for locally compact Q-manifolds ([24], [21], [47]).

The main reason that Q-manifolds were introduced to geometric topology is the following theorem:

Theorem 3.13 ([23], [27])**.** *A homotopy equivalence $f : X \to Y$ between compact ANR's is simple (i.e., $[f] = 0 \in Wh(X)$) iff $f\times id_Q$ is homotopic to a homeomorphism.*

This theorem gives a positive answer to Whitehead's Homeomorphism Conjecture.

Theorem 3.14 (The topological invariance of the Whitehead torsion)**.** *Let $f : K \to L$ be a homeomorphism between compact polyhedra. Then $[f] = 0 \in Wh(L)$.*

Also, in [26] there the following triangulation result for Q-manifolds.

Theorem 3.15 (Triangulation of Q-manifolds)**.** *Let M be a locally compact Q-manifold. Then there is a locally compact polyhedron K such that M is homeomorphic to $L\times Q$.*

Combining the above theorem with the basic facts on Q-manifolds we get a positive answer to Borsuk's question.

Theorem 3.16 ([23], [95])**.** *Let X be a compact ANR. Then for each $\varepsilon > 0$ there is a finite CW-complex K_ε that it is ε-homotopy equivalent to X with control in X.*

Notice that in general given X and $\varepsilon > 0$ a construction of a finite CW-complex that ε-dominates X is not hard. The complex is constructed using nerves of open covers of X.

Now we can go back to the proof of the Fundamental Theorem of K-theory. The proof presented is based on the work in [62] and [72]. We can assume that X is a compact Q-manifold and an element of $Wh(X\times S^1)$ is represented by a strong deformation retraction $f : Y \to X\times S^1$ with Y a compact Q-manifold. Form the pull-back:

$$\begin{array}{ccc} \bar{Y} & \xrightarrow{\bar{f}} & X\times\mathbb{R} \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f} & X\times S^1 \end{array}$$

Denote by ζ the covering translation on $\bar{Y}$ that corresponds to $+1$ action on the reals. For a subset $A \subset \mathbb{R}$ we write $Y_A = \bar{f}^{-1}(X\times A)$.

Let $K_\pm$ be compact sub-ANR's that dominate $Y_\pm$, relY_0, where $Y_- = \bar{Y}_{(-\infty,0]}$, $Y_+ = \bar{Y}_{[0,\infty)}$. The existence of such complexes follows from the fact that $\bar{f}$ is a bounded (over $\mathbb{R}$) strong deformation retraction ([20], [49]). Write

$$Y_\pm \xrightarrow{d_\pm} K_\pm \xrightarrow{u_\pm} Y_\pm$$

for the dominations. Write $K = K_- \cup_{Y_0} K_+$, and get the maps

$$\bar{Y} \xrightarrow{d} K \xrightarrow{u} \bar{Y},$$

induced from the maps of the dominations. Each of the spaces $K_\pm$, K,, Y and $Y_\pm$ contains a copy of X times an interval containing 0. Let $L_\pm$, L, Z and $Z_\pm$ be the spaces formed from the ones above by squeezing $X\times$interval to a single copy of X. Then the domination maps induce maps, denoted the same, on the new spaces. Set

- $z : L \xrightarrow{u} Z \xrightarrow{\zeta^{-1}} Z \xrightarrow{d} L.$.
- $f_+ : L_+ \xrightarrow{u} Z_+ \xrightarrow{\zeta} Z_+ \xrightarrow{d} L_+.$.
- $f_- : L_- \xrightarrow{u} Z_- \xrightarrow{\zeta^{-1}} Z_- \xrightarrow{d} L_-.$.

3.1.1. *Relaxation.* We start by recalling the construction of the relaxation of f ([22], [24], [59], [62], [63], [80]). This construction is the map

$$Wh(X\times S^1) \to Wh(X\times S^1, p)_c$$

that splits the forget control map.

Let $q : X\times\mathbb{R} \to \mathbb{R}$ be the projection. Remember that the strong deformation retraction $\bar{f}$ is bounded over $\mathbb{R}$ i.e., it is a $q^{-1}(k)$-homotopy equivalence, for some $k > 0$. The number k depends on how many times the image of f wraps around S^1. Let

$$\phi_d : X\times\mathbb{R} \to X\times\mathbb{R}, \;\; \phi_d(x,r) = (x, r/d).$$

be the homeomorphism that shrinks distances in the $\mathbb{R}$-direction. Then the composition $\phi_d \circ \bar{f}$ is a $q^{-1}(k/d)$-strong deformation retraction over $\mathbb{R}$. By choosing large enough d and the Squeezing Theorem, we can assume that $\bar{f}$ is a controlled (over $\mathbb{R}$) strong deformation retraction. Thus, the composition

$$\bar{Y} \xrightarrow{\bar{f}} X \times \mathbb{R} \to \mathbb{R}$$

is an approximate fibration (Theorem 3.7).

Now we need the approximate isotopy lifting property for approximate fibrations. Remember that an isotopy of X is a homeomorphism $g : B \times [0,1] \to B \times [0,1]$ which preserves the second coordinate. If $p : E \to B$ is an approximate fibration, the approximate isotopy lifting property for p means that, given an isotopy g of the base space and a $\varepsilon > 0$ there is an isotopy G of E making the diagram ε-commutative:

$$\begin{array}{ccc} E \times [0,1] & \xrightarrow{G} & E \times [0,1] \\ {\scriptstyle p \times \mathrm{id}_{[0,1]}} \downarrow & & \downarrow {\scriptstyle p \times \mathrm{id}_{[0,1]}} \\ B \times [0,1] & \xrightarrow{g} & B \times [0,1] \end{array}$$

More precisely,

$$d(p \times \mathrm{id}_{[0,1]}(G(e,s)), g(p(e),t)) < \varepsilon, \text{ for all } (e,t) \in E \times [0,1].$$

If both X and E are finite dimensional or Q-manifolds, then approximate fibrations have the approximate isotopy lifting property ([60], [61], [64], [63]).

In our setting, let $g : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ be the PL isotopy (i.e., a homeomorphism that preserves the second coordinate) such that

(1) g is supported on $[-1,3] \times [0,1]$, that is g is that identity outside the given set, and
(2) for each $s \in [0,1]$, g_s takes $[-1,-0.5]$ linearly onto $[-1, s-0.5]$, takes $[-0.5, 1.5]$ linearly onto $[s-0.5, s+1.5]$ and takes $[1.5, 3]$ linearly onto $[s+1.5, 3]$.

In particular, $g_1|[-0.5, 1.5]$ has the form $t \mapsto t+1$. By choosing a small $\varepsilon > 0$ and applying the approximate isotopy lifting property, we get an isotopy:

$$h : \bar{Y} \times [0,1] \to \bar{Y} \times [0,1],$$

with compact support, from the identity such that

$$h_1(Y_{(-\infty,0)}) \supset Y_{(-\infty,1]}.$$

Define $\hat{Y} = \zeta h_1(Y_{(-\infty,0]}) \setminus Y_{(-\infty,0)}/\!\sim$, where a point $x \in Y_0$ is identified with $\zeta h_1(x) \in \zeta h_1(Y_0)$. There is a strong deformation retraction:

$$\hat{f} : \hat{Y} \to X \times S^1$$

Then $\hat{f} \in Wh(X\times S^1, p)_c$. The reason is that in the infinite cyclic cover induced by $\hat{f}$ the fundamental domain is 'larger' and that makes $\hat{f}$ more controlled over S^1. The fact that it is controlled follows from the Sucking Principle.

3.1.2. *Maps to Nil-groups.* We describe the Nil-groups as summands of the Whitehead group:

(1) $j_\pm : \widetilde{\mathrm{Nil}}(X) \to Wh(X\times S^1)$. The two injections are given by taking mapping tori:
   (a) $j_+(Y, g) = [T'(g), X\times S^1]$, where $T'(g) = Y\times[0,1]/(g(y),1) \sim (y,0)$.
   (b) $j_-(Y, f) = [T(g), X\times S^1]$, where $T(g) = Y\times[0,1]/(g(y),0) \sim (y,1)$.

(2) The projections are constructed by looking at the translation map on $\bar{Y}$ and Z. More precisely,
$$p_\pm : Wh(X\times S^1) \to \widetilde{\mathrm{Nil}}(X), \quad p_\pm(f) = [L_\pm, f_\pm].$$

Since K retracts to X, L also retracts to X and the pair $(T(z), X\times S^1) = 0 \in Wh(X\times S^1)$. In the mapping torus $T(z)$ (Figure 1):

(1) There is an obvious embedding of Y if we follow the restriction of the mapping torus to Y_0.
(2) A variation of the previous argument is used to construct an embedding of $\hat{Y}$, disjoint from Y into the part of $T(z)$ in L_+.
(3) The restriction, N_-, of the mapping torus to L_- represents the image of f under the composition:
$$Wh(X\times S^1) \xrightarrow{p_-} \widetilde{\mathrm{Nil}}(X) \xrightarrow{j_-} Wh(X\times S^1).$$
More precise, there is a strong deformation retraction $f_- : N_- \to X\times S^1$ such that $j_-\circ p_-([f]) = [f_-]$.
(4) Let N_+ be the region in $T(z)$ between Y and $\bar{Y}$, considered "backwards" i.e., with the reverse orientation of that given by the mapping torus. Then N_+ represents the image of the element $[f]$ under the map.
$$Wh(X\times S^1) \xrightarrow{p_+} \widetilde{\mathrm{Nil}}(X) \xrightarrow{j_+} Wh(X\times S^1).$$
(5) The rest of the region of $T(z)$, W has the property that it is homeomorphic to the mapping torus of a retraction and thus, there is a strong deformation retraction $r : W \to X\times S^1$ which has zero torsion in $Wh(X\times S^1)$

Combining the above observations and the sum theorem for the Whitehead torsion we get that, in $Wh(X\times S^1)$:
$$0 = (T(z), X\times S^1) = [f_-] - [f] + [f_+] - [\bar{f}] + [r]$$
and thus
$$[f] = j_-\circ p_-([f]) + j_+\circ p_+([f]) - [\bar{f}].$$
The last equation provides the splitting of $[f]$ into the various components. Figure 1 gives a presentation of the spaces used in the proof. It should be considered as the geometric analogue of Ranicki's pentagon ([77], p. 95, p. 99).

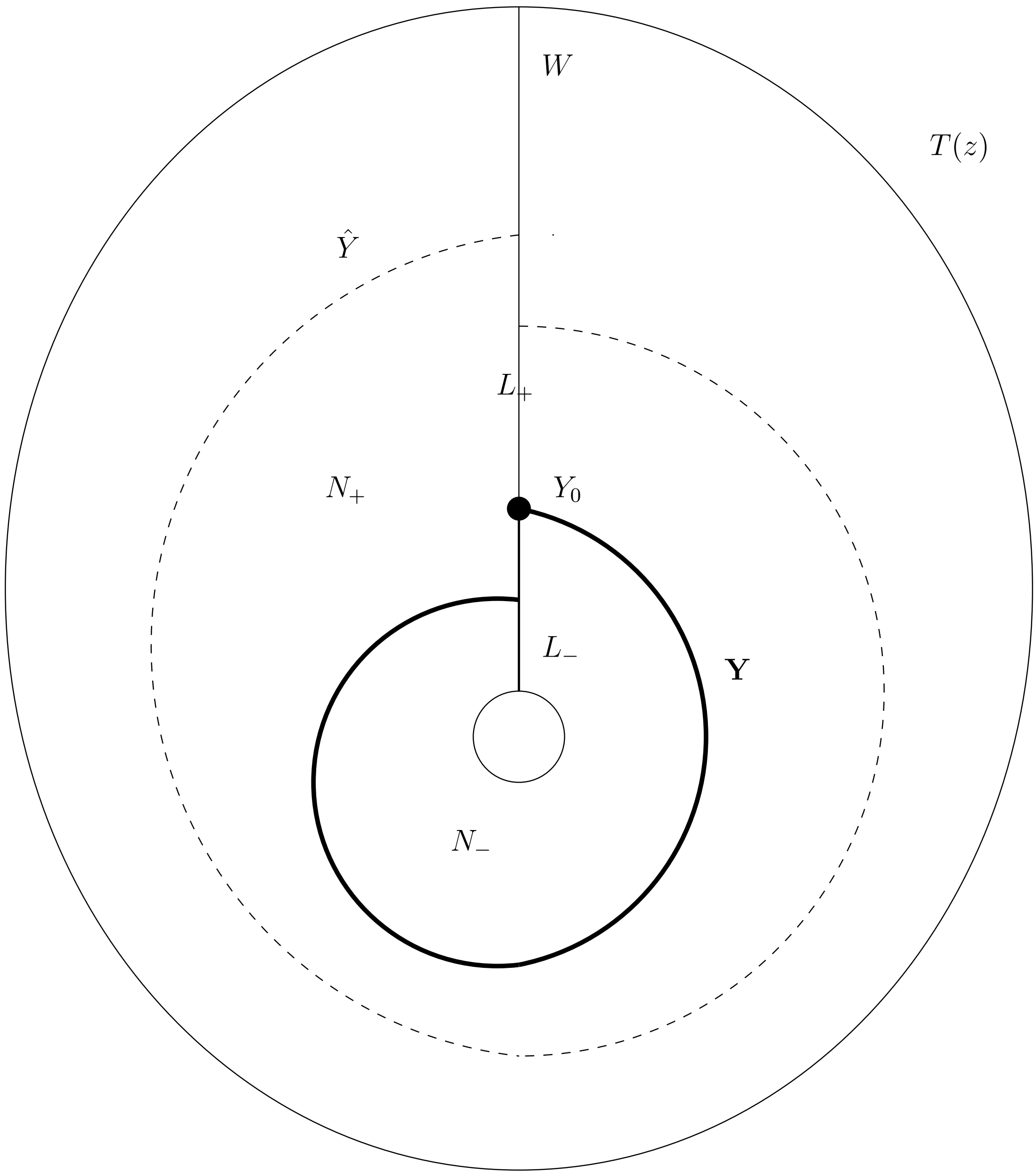

FIGURE 1. The mapping Torus of the Translation

3.2. **Waldhausen's Splitting Theorem Revisited.** There is a controlled interpretation of Waldhausen's Theorem. Let H be a subgroup of $G_0 \cap G_1$ and $\Gamma = G_0 *_H G_1$. Set

$$M = M(\iota_0) \cup_{BH} M(\iota_1)$$

where $\iota_i : BH \to BG_i$ be the inclusion map of classifying spaces. Van Kampen's Theorem implies that $\pi_1(M) = \Gamma$.

Theorem 3.17 (Waldhausen's Theorem)**.** *There is a splitting*

$$Wh(M) = Wh(M,p)_c \oplus \widetilde{Nil}(\mathbb{Z}H; \mathbb{Z}[G_0 \backslash H], \mathbb{Z}[G_1 \backslash H]),$$

where $p : M \to [-1,1]$ is the natural projection map to the second coordinate of the mapping cylinders. Furthermore, there is an exact sequence

$$Wh(H) \to Wh(G_0) \oplus Wh(G_1) \to Wh(M,p) \to \tilde{K}_0(\mathbb{Z}H) \to \tilde{K}_0(\mathbb{Z}G_0) \oplus \tilde{K}_0(\mathbb{Z}G_1) \to \dots$$

For this interpretation of Waldhausen's Theorem, see [70].

Question. Is there a geometric map that describes the splitting of the forget control map? More precisely, is there a geometric relaxation map:

$$Wh(M) \to Wh(M,p)_c$$

as in the case over the circle?

Remark 3.18. There is the analogue of the "infinite cyclic" cover of M is this case. The natural action on the infinite cyclic cover is of D_∞, the infinite dihedral group. The answer to the question seems to involve a version of $\mathbb{Z}/2\mathbb{Z}$-equivariant relaxation construction

3.3. **Geometric Applications of Nil-groups.** Nil-groups appear as obstructions to splitting theorems. Farrell's fibering theorem is such a result.

Theorem 3.19 (Farrell ([38], Siebenmann [80]))**.** *Let M^n, $n \geq 6$, be a closed connected manifold and $p : M \to S^1$ a map such that:*

1. *$p_* : \pi_1(M) \to \pi_1(S^1)$ is an epimorphism with kernel G. Thus $\pi_1(M) = G \rtimes_\alpha \mathbb{Z}$.*
2. *The infinite cyclic cover of M induced by p is finitely dominated.*

Then

1. *There is an obstruction $n(f) \in \widetilde{Nil}(\mathbb{Z}G, \alpha)$ which vanishes iff f is homotopic to an approximate fibration.*
2. *If (1) is satisfied, then there is an obstruction $\tau(f) \in Wh(M,p)_c$ which vanishes iff f is homotopic to a fiber bundle.*

Remark 3.20.

1. Farrell's obstructions are encoded as a total obstruction geometrically in [80] and algebraically in [77]. Essentially, the obstruction is described as follows: Let $\bar{M}$ be the infinite cyclic cover of M and ζ a generating cover translation. Then, the total obstructions is defined as:
$$\Phi(f) = \tau(T(\zeta) \to M) \in Wh(G \rtimes_\alpha \mathbb{Z}).$$
Because $T(\zeta)$ is not a finite complex, for the exact meaning of the torsion see [77].

(2) Originally, Farrell's theorem was not stated using controlled topology. To describe the classical setting, using transversality, first homotope p so that the fundamental domain of the infinite cyclic cover is a cobordism W. The first obstruction was

$$\mathbf{n}(f) = n(f) + k(f) \in \widetilde{\mathrm{Nil}}(\mathbb{Z}G, \alpha) \oplus \tilde{K}_0(\mathbb{Z}G)$$

that vanishes iff W is an h-cobordism. Once $\mathbf{n}(f)$ vanishes, the second obstruction

$$\tau(f) = \tau(W) \in Wh(G)_{\alpha_*} = Wh(G)/\{x - \alpha_*(x) : \ x \in Wh(G)\}$$

is the torsion of the h-cobordism W. Then $\tau(f)$ vanishes iff p is homotopic to a fiber bundle.

(3) Notice that $Wh(M, p)_c$ fits into an exact sequence ([70]):

$$Wh(G) \xrightarrow{1-\alpha_*} Wh(G) \to Wh(M, p)_c \to \tilde{K}_0(\mathbb{Z}G) \xrightarrow{1-\alpha_*} \tilde{K}_0(\mathbb{Z}G)$$

(4) The controlled version of Part (1) of the Theorem is stated implicitly in different papers ([22], [62], [63]).

A more general splitting theorem is given in [40]. Let M^n, $n \geq 6$, be a closed connected manifold with $\pi_1(M) = G \rtimes_\alpha \mathbb{Z}$ ([42] for the algebraic version). Let $f : M \to M'$ be a homotopy equivalence. Also, let N' be a codimension 1 submanifold of M' such that $\pi_1(N') = G$. The map f is called *splittable* along N' if f is homotopic to a map g which is a homotopy equivalence of pairs:

$$g : (M, N) \to (M', N'),$$

where N is a codimansion 1 submanifold of M.

Theorem 3.21 ([40])**.** *Under the above notation, f is splittable along N' iff*

$$\mathbf{n}(f) = p(\tau(f)) \in \widetilde{Nil}(\mathbb{Z}G, \alpha) \oplus \tilde{K}_0(\mathbb{Z}G)$$

vanishes.

For the Waldhausen's splitting, let M^n, $n \geq 6$, be a closed connected manifold and $N \subset M$ a codimension 1 connected submanifold with trivial normal bundle such that $M \setminus N$ has two components. Thus $\pi_1(M) = G_0 *_H G_1$ where $\pi_1(N) = H$. Assume that H is *square root closed* in G_i (i.e., if $g^2 \in H$ then $g \in H$). Let $f : W^n \to M^n$ be a homotopy equivalence. Then f is called *splittable* along N if f is homotopic to g so that there is a codimension 1 submanifold $V = g^{-1}(N)$ so that the restrictions g is a homotopy equivalence of pairs

$$g : (W, V) \to (M, N).$$

Theorem 3.22 ([17])**.** *In the above setting f is splittable along N if the component*

$$\mathbf{n}(f) = p(\tau(f)) \in \widetilde{Nil}(\mathbb{Z}H, \mathbb{Z}[G_0 \setminus H], \mathbb{Z}[G_1 \setminus H]) \oplus ker(\tilde{K}_0(\mathbb{Z}H) \to \tilde{K}_0(\mathbb{Z}G_1) \oplus \tilde{K}_0(\mathbb{Z}G_2))$$

vanishes.

Remark 3.23.

(1) The condition that H is square root closed in both groups is needed for the vanishing of the UNil-groups, the surgery theory Nil-groups ([16]).
(2) In [17], a splitting theorem is given when $M \setminus N$ is a single component i.e., when $\pi_1(M)$ is an HNN-extension of H.

3.4. **Remarks on Nil-groups.** From the above discussion, it follows that the various Nil-groups are the obstructions for K-theory to be a homology theory. They measure the failure of K-theory to satisfy excision. The component of K-theory that satisfies homological properties, like excision, is given by the controlled component. In other words, algebraically, the controlled K-groups can be defined as the part of K-theory that satisfies excision. We will make two more remarks emphasizing these properties of the Nil-groups.

3.4.1. *Categorical Double Mapping Cylinders.* First, we specialize to amalgamated free products. K-theory can be defined for symmetric monoidal categories in general. Then $K_*(R) = K_*(\mathcal{F}_R^{Iso})$ where $\mathcal{F}_R^{Iso}$ is the category of finitely generated free (left) R-modules with morphisms R-isomorphisms. Let

$$\begin{array}{ccc} R & \longrightarrow & A_0 \\ \big\downarrow & & \big\downarrow \\ A_1 & \longrightarrow & \Lambda \end{array}$$

be a push out diagram of rings. Let $\mathcal{P}$ be the double mapping cylinder (the homotopy push-out) in the category of symmetric monoidal categories

$$\begin{array}{ccc} \mathcal{F}_R^{Iso} & \longrightarrow & \mathcal{F}_{A_0}^{Iso} \\ \big\downarrow & & \big\downarrow \\ \mathcal{F}_{A_1}^{Iso} & \longrightarrow & \mathcal{P} \end{array}$$

defined in [88], Construction 5.1. The above diagram induces a homotopy cartesian diagram of spectra ([88]):

$$\begin{array}{ccc} \mathbb{K}.(R) & \longrightarrow & \mathbb{K}.(A_0) \\ \big\downarrow & & \big\downarrow \\ \mathbb{K}.(A_1) & \longrightarrow & \mathbb{K}.(\mathcal{P}) \end{array}$$

Thus the K-theory of $\mathcal{P}$ is the 'controlled K-theory with control over the interval' (Section 3.2). Also, the universal properties of the double mapping cylinder imply that there is a natural functor

$$F : \mathcal{P} \to \mathcal{F}_\Lambda^{Iso}.$$

Waldhausen's Splitting Theorem implies that:

$$\widetilde{\mathrm{Nil}}_{i-1}(R; B_0, B_1) \cong \mathrm{Coker}(F_* : K_i(\mathcal{P}) \to K_i(\Lambda)).$$

where $A_i = R \oplus B_i$ as R-bimodules, for $i = 0, 1$.

3.4.2. *Continuous Control K-theory.* There is a connection between Nil-groups and continuously controlled K-groups. The definitions and the main calculations for the controlled K-groups are given in [1].

There are two cases to be consider. First we assume that $p : X \times S^1 \to S^1$ is the projection map and X is a finite CW-complex (or a compact ANR). Let $M(p)$ denote the mapping cylinder of p. Then there is an exact sequence ([2]):

$$Wh(X \times S^1, p)_c \to Wh(X \times S^1) \to Wh(M(p), S^1)_{cc} \to \tilde{K}_0(X \times S^1, p)_c \to \tilde{K}_0(X \times S^1)$$

where $Wh(M(p), S^1)_{cc}$ is the continuously controlled Whitehead group. Thus, if we combine this with this exact sequence with the Fundamental Theorem of Algebraic K theory, we get ([70]):

$$Wh(M(p), S^1)_{cc} \cong \widetilde{Nil}(X) \oplus \widetilde{Nil}(X).$$

Let $\Gamma = G_1 *_{G_0} G_2$ be an amalgamated free product. Consider the induced maps $\phi_i : BG_0 \to BG_i$, $i = 1, 2$. Let X be the double mapping cylinder:

$$X = M(\phi_1) \cup_{BG_0} M(\phi_1).$$

Then $\pi(X) = \Gamma$ and X comes with a natural map $\rho : X \to I$ to the unit interval. Then, as before we have an exact sequence:

$$Wh(X, \rho)_c \to Wh(X) \to Wh(M(\rho), I)_{cc} \to \tilde{K}_0(X, \rho)_c \to \tilde{K}_0(X)$$

Using Waldhausen's splitting theorem we get ([70]):

$$Wh(M(\rho), I)_{cc} \cong \widetilde{Nil}(\mathbb{Z}G_0, \mathbb{Z}[G_1 \backslash G_0], \mathbb{Z}[G_1 \backslash G_0]).$$

Remark 3.24. In [2], the continuously controlled groups are defined for holink triples. Our interpretation is equivalent to that one in the above cases and it is taken from [1].

4. Controlled Topology and the Isomorphism Conjecture

In this section, we describe the controlled aspects of the Farrell–Jones Isomorphism Conjecture ([43]). Most of the material is contained in [34]. Let Γ be a discrete group and $\mathcal{C}_\Gamma$ a class of subgroups of Γ (i.e. a collection of subgroups of Γ closed under taking conjugates and subgroups). The *classifying space* of the class $\mathcal{C}_\Gamma$, $\mathcal{E}\mathcal{C}_\Gamma$ is the Γ-complex whose isotropy groups are in $\mathcal{C}_\Gamma$ and its non-empty fixed point sets are contractible. A model for this space, reminiscent of the bar construction ([43]), is given as follows: It is the realization of a semi-simplicial complex with n-simplex given by a sequence

$$\sigma = \gamma_0 \Gamma_0 (\gamma_1 \Gamma_1, \gamma_2 \Gamma_2, \ldots, \gamma_n \Gamma_n)$$

with $\Gamma_i \in \mathcal{C}$ such that $\gamma_i^{-1}\Gamma_{i-1}\gamma_i \subset \Gamma_i$ for $i = 1, \ldots, n$. The face operator is given by

$$\partial_i\sigma = \begin{cases} \gamma_0\gamma_1\Gamma_1(\gamma_2\Gamma_1, \ldots, \gamma_n\Gamma_n), & i = 0, \\ \gamma_0\Gamma_0(\gamma_1\Gamma_1, \ldots, \gamma_{i-1}\Gamma_{i-1}, \gamma_i\gamma_{i+1}\Gamma_{i+1}, \gamma_{i+1}\Gamma_{i+1}, \ldots, \gamma_n\Gamma_n), & 0 < i < n, \\ \gamma_0\Gamma_0(\gamma_1\Gamma_1, \ldots, \gamma_{n-1}\Gamma_{n-1}), & i = n. \end{cases}$$

The group Γ acts on $E\mathcal{C}_\Gamma$ by

$$\gamma\sigma = (\gamma\gamma_0)\Gamma_0(\gamma_1\Gamma_1, \gamma_2\Gamma_2, \ldots, \gamma_n\Gamma_n), \text{ for } \gamma \in \Gamma.$$

We write $\mathcal{BC}_\Gamma$ for the orbit space $\mathcal{EC}_\Gamma/\Gamma$. To ensure that $\mathcal{BC}_\Gamma$ is a simplicial complex, we subdivide $\mathcal{EC}_\Gamma$ twice, i.e. $\mathcal{BC}_\Gamma = (\mathcal{EC}_\Gamma)''/\Gamma$.

The construction of the classifying space is functorial with respect to group homomorphisms. Let $\mathcal{C}_\Gamma$ be a class of subgroups of Γ. For if $\rho : \Gamma \to G$ be a group homomorphism then ρ induces a ρ-equivariant map

$$\bar{\rho} : \mathcal{EC}_\Gamma \to \mathcal{EC}_G,\ \gamma_0\Gamma_0(\gamma_1\Gamma_1, \gamma_2\Gamma_2, \ldots, \gamma_n\Gamma_n) \mapsto \rho(\gamma_0)\rho(\Gamma_0)(\rho(\gamma_1)\rho(\Gamma_1), \rho(\gamma_2)\rho(\Gamma_2), \ldots, \rho(\gamma_n)\rho(\Gamma_n))$$

where $\mathcal{C}_G$ is a class of subgroups of G that contains the images, under ρ, of the elements of $\mathcal{C}_\Gamma$. The map $\bar{\rho}$ induces a map ρ' to the quotient spaces.

For each simplicial complex K, we write $\text{cat}(K)$ for the category of simplices of K, viewed as a partially ordered set. Thus, objects are the simplices of K and there is a single morphism from σ to τ whenever $\sigma \leq \tau$.

Definition 4.1 ([3])**.** Let $p : E \to B$ be a map with $B = |K|$, the geometric realization of a simplicial complex. The map is said to have a *homotopy colimit structure* if there is a functor

$$F : \text{cat}(K)^{op} \to \textbf{Top}$$

such that:

- $E = \text{hocolim}_{\text{cat}(K)^{op}}(F)$.
- $p = \text{hocolim}_{\text{cat}(K)^{op}}(\nu)$, where ν is the natural transformation from F to the constant point functor.

Remark 4.2. For a simplicial complex K and a functor $F : \text{cat}(K)^{op} \to \textbf{Top}$, the homotopy colimit is defined as:

$$\text{hocolim}_{\text{cat}(K)^{op}}(F) = \coprod_{\sigma \in \text{cat}(K)^{op}} F(\sigma) \times |\sigma| / \sim$$

where for simplices $\tau \geq \sigma$, $x \in F(\tau)$, $t \in |\sigma|$,

$$(x, t) \sim (F(\tau \geq \sigma)(x), t).$$

Notice that $(x, t) \in F(\tau) \times |\sigma| \subset F(\tau) \times |\tau|$ and $F(\tau \geq \sigma) : F(\tau) \to F(\sigma)$.

Notation. Let $p : E \to B$ be a map, $B = |K|$ and $\mathrm{cat}(K)^{op}$ the category of the simplicial complex K. In our setting, there is the *barycentre functor*

$$\mathrm{bar}(p) : \mathrm{cat}(K)^{op} \to \mathbf{Top},\ \sigma \mapsto p^{-1}(\hat{\sigma})$$

where $\hat{\sigma}$ is the barycentre of σ. More precisely, for maps that we will use, for each $\tau \geq \sigma$, there is a continuous map $p^{-1}(\hat{\tau}) \to p^{-1}(\hat{\sigma})$.

Remark 4.3. We give basic examples of maps that admit a homotopy colimit structure. The proofs follow from direct calculations (also [87]).

(1) Let Γ be a discrete group, $E\Gamma$ a free contractible Γ-complex and $\mathcal{E}\mathcal{C}_\Gamma$ the classifying complex for the family of subgroups of Γ. Let

$$p_\Gamma : E\Gamma \times_\Gamma \mathcal{E}\mathcal{C}_\Gamma \to \mathcal{E}\mathcal{C}_\Gamma/\Gamma = \mathcal{B}\mathcal{C}_\Gamma$$

be the projection map to the second coordinate. Then p_Γ has a homotopy colimit structure with respect to the barycenter functor $\mathrm{bar}(p_\Gamma)$. Notice that, in this case, $p_\Gamma^{-1}(\hat{\sigma})$ is a space of type $B\Gamma_\sigma$, where Γ_σ is the isotropy group of σ, an element in the class $\mathcal{C}_\Gamma$.

(2) Let $\rho : \Gamma \to G$ be a group epimorphism. Then the map

$$q : E\Gamma \times_\Gamma \mathcal{E}\mathcal{C}_\Gamma \xrightarrow{p_\Gamma} \mathcal{B}\mathcal{C}_\Gamma \xrightarrow{\rho'} \mathcal{B}\mathcal{C}_G$$

has a homotopy colimit structure with respect to the functor $\mathrm{bar}(q)$, where $\mathcal{C}_G = \rho(\mathcal{C}_\Gamma)$, the class of subgroups of G consisting of the images of elements of $\mathcal{C}_\Gamma$.

Let $F : \mathcal{C} \to \mathcal{D}$ and $X : \mathcal{C} \to \mathbf{Top}$ be two functors. Then *Segal's Pushdown Construction* (see for example [58]) defines a functor $F_*X : \mathcal{D} \to \mathbf{Top}$ such that

$$\mathrm{hocolim}_{\mathcal{C}} X \simeq \mathrm{hocolim}_{\mathcal{D}} F_*X.$$

We will explicitly describe the construction to the case of Part(2) in Remark 4.3. In this case, we start with a map:

$$q : E\Gamma \times_\Gamma \mathcal{E}\mathcal{C}_\Gamma \xrightarrow{p_\Gamma} \mathcal{B}\mathcal{C}_\Gamma \xrightarrow{\rho'} \mathcal{B}\mathcal{C}_G$$

The map ρ' induces a functor

$$P : \mathrm{cat}(\mathcal{B}\mathcal{C}_\Gamma)^{op} \to \mathrm{cat}(\mathcal{B}\mathcal{C}_G)^{op}$$

We will describe the functor

$$P_*\mathrm{bar}(p) : \mathrm{cat}(\mathcal{B}\mathcal{C}_G)^{op} \to \mathbf{Top}.$$

For each simplex σ of $\mathcal{B}\mathcal{C}_G$, let $P{\downarrow}\sigma$ be the over category. In this case, the objects of $P{\downarrow}\sigma$ are simplices τ of $\mathcal{B}\mathcal{C}_\Gamma$ such that $\rho'(\tau)$ contains σ as a face. Set

$$p_\sigma = p| : p^{-1}(|P{\downarrow}\sigma|) \to |P{\downarrow}\sigma|.$$

Then $P_*\mathrm{bar}(p)(\sigma) = \mathrm{hocolim}_{P\downarrow\sigma}\mathrm{bar}(p_\sigma)$. Summarizing:

Proposition 4.4. *There is a homotopy equivalence:*

$$hocolim_{cat(\mathcal{BC}_\Gamma)^{op}}(bar(p_\Gamma)) \simeq hocolim_{cat(\mathcal{BC}_G)^{op}}(P_* bar(p_\Gamma)).$$

Remark 4.5. Segal's pushdown construction it is the categorical analogue of the 'change of control' map. Essentially, for a map $p : X \to Y$ it shows how control over X can be computed by computing the control over Y with 'coefficients' the contribution of the fibers of p.

The class of subgroups of Γ of interest in the Isomorphism Conjecture is the class of *virtually cyclic subgroups*, denoted $\mathcal{G}_\Gamma$. They split into two categories:

- Finite subgroups of Γ.
- Virtually infinite cyclic subgroups of Γ i.e., subgroups which contain an infinite cyclic subgroup of finite index.

The subgroups of the second type are two-ended subgroups of Γ ([36]) and they split into two types:

- Groups H that admit an epimorphism to $\mathbb{Z}$ with finite kernel i.e.
$$H \cong K \rtimes \mathbb{Z},$$
with K finite.
- Groups H that admit an epimorphism to the infinite dihedral subgroup D_∞ with finite kernel, i.e.
$$H \cong A *_B C, \quad [B : 1] < \infty, \ [A : B] = [C : B] = 2.$$

Let $\mathbb{S}pectra$ denote the category of spectra with morphisms *strict maps* i.e., maps between spectra that commute with the bonding maps. Let

$$\mathbb{S} : \mathbf{Top} \to \mathbb{S}pectra$$

is a *homotopy invariant functor* i.e., a functor that maps homotopy equivalent spaces to homotopy equivalent spectra. Let Γ be a discrete group As before, let $\mathcal{G}_\Gamma$ be the class of virtually cyclic (finite or infinite) subgroups of Γ. Let $\mathcal{EG}_\Gamma$ be the classifying Γ-complex for the class $\mathcal{G}_\Gamma$ and

$$p_\Gamma : E\Gamma \times_\Gamma \mathcal{EG}_\Gamma \to \mathcal{EG}_\Gamma/\Gamma = \mathcal{BG}_\Gamma$$

be the projection map. Let $\text{cat}(\mathcal{BG}_\Gamma)^{op}$ be the category corresponding to the partially ordered set of simplices of $\mathcal{BG}_\Gamma$. Let $r : Y \to B\Gamma$ be a bundle. Form the pull-back:

$$\begin{array}{ccc} \bar{Y} & \longrightarrow & Y \\ {\scriptstyle\rho}\downarrow & & \downarrow{\scriptstyle r} \\ E\Gamma \times_\Gamma \mathcal{EG}_\Gamma & \longrightarrow & B\Gamma \\ {\scriptstyle p_\Gamma}\downarrow & & \downarrow \\ \mathcal{BG}_\Gamma & \longrightarrow & * \end{array}$$

Notice that

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_\Gamma)^{op}}\mathrm{bar}(p_\Gamma \circ \rho) \cong \bar{Y} \simeq Y$$

Let $*$ denote the category with a single object and a single morphism. Let

$$F_\Gamma : * \to \mathbf{Top}, \quad F_\Gamma(*) = Y$$

The $\mathbb{S}$-Bundle Isomorphism Conjecture. With the notation above, the functor from $\mathrm{cat}(\mathcal{BG}_\Gamma)^{op}$ to $*$, induces a homotopy equivalence of spectra:

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_\Gamma)^{op}}\mathbb{S}\circ\mathrm{bar}(p_\Gamma \circ \rho) \to \mathrm{hocolim}_{\mathrm{cat}(*)}\mathbb{S}\circ\mathrm{bar}(F_\Gamma) \cong \mathbb{S}(Y).$$

Remark 4.6.

(1) In the classical setting the homotopy colimit was written as homology:

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_\Gamma)^{op}}\mathbb{S}\circ\mathrm{bar}(p_\Gamma \circ \rho) = \mathbb{H}.(\mathcal{BG}_\Gamma, \mathbb{S}(p_\Gamma \circ \rho)).$$

The right hand side is Quinn homology. We will use both notations.

(2) Homotopy colimits were used in the formulation of the assembly map in [35] and [56]. Actually in [56], all the known constructions of the assembly are compared.

(3) The original fiber isomorphism conjecture was stated under the assumption that r is a fibration ([43]). In that more general setting, Quinn homology is used in the formulation.

(4) If a torsion free group satisfies the $\mathbb{K}$-IC then it satisfies the Vanishing Conjectures for the Whitehead group. or a complete account of the connections between the Isomorphism Conjectures, Conjectures 3 and 4 and other conjectures in algebra is given in [11].

4.1. **Spaces over the Circle.** Let $\Gamma = G \rtimes_\alpha \mathbb{Z}$. Here α is the automorphism of G induced by the action of the generator of $\mathbb{Z}$. The automorphism α is well-defined up to inner automorphisms. By choosing a suitable right G-space for EG, there is an α-equivariant homeomorphism

$$\psi : EG \to EG$$

i.e., $\psi(xg) = \psi(x)\alpha(g)$. By taking quotients, we see that there is a homeomorphism

$$\phi : BG \to BG$$

that induces the map α in the fundamental group, again up to inner automorphisms. Choose as a model for $B\Gamma$ the mapping torus of ϕ:

$$B\Gamma = BG \times [0,1]/\sim, \ (\phi(x), 0) \sim (x, 1).$$

Then a model for $E\Gamma$ is the infinite mapping telescope of ψ:

$$E\Gamma = EG \times [0,1] \times \mathbb{Z}/\sim, \quad (x, 1, n) \sim (\psi(x), 0, n+1).$$

The right action of Γ on $E\Gamma$ is given by:

$$(x,t,n)(g,m) = (x\alpha^n(g), t, n+m).$$

The map

$$\Phi : BG \times [0,1] \to B\Gamma,\ \Phi(x,t) = [x,t]$$

has the property that:

$$\Phi(x,0) = [x,0], \quad \Phi(x,1) = [x,1] = [\phi(x),0]$$

i.e., it defines a homotopy between the identity map and the map ϕ on BG inside $B\Gamma$. Also, the natural projection map to the second coordinate $\rho : B\Gamma \to S^1$ is a bundle. Consider the commutative diagram

$$\begin{array}{ccc} E\Gamma\times_\Gamma \mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & E\Gamma\times_\Gamma \mathcal{E}\mathcal{G}_\Gamma \\ {\scriptstyle q_\Gamma}\downarrow & & \downarrow{\scriptstyle p_\Gamma} \\ S^1\times \mathcal{B}\mathcal{G}_\Gamma & \longrightarrow & \mathcal{B}\mathcal{G}_\Gamma \end{array}$$

where q_Γ is the composite $E\Gamma \to B\Gamma \xrightarrow{\rho} S^1$ is the first coordinate and the projection $\mathcal{E}\mathcal{G}_\Gamma \to \mathcal{B}\mathcal{G}_\Gamma$ in the second. In the applications the spectrum $\mathbb{S}$ has an infinite loop space structure. Let $\mathcal{CF}$ be the homotopy cofiber:

$$\mathbb{S}(BG) \xrightarrow{1-\phi_*} \mathbb{S}(BG) \to \mathcal{CF}$$

The projection map:

$$BG \times [0,1] \to T(\phi) = B\Gamma$$

induces a homotopy between two maps from BG to $B\Gamma$. The first map is

$$BG \to B\Gamma,\ \ x \mapsto (x,0).$$

The other map is

$$BG \to B\Gamma,\ \ x \mapsto (x,1) = (\phi(x),0).$$

Thus the maps id_{BG} and ϕ are homotopic in $B\Gamma$. That mean the map $1-\phi_*$ is null homotopic. Therefore, there is an induced map:

$$f : \mathcal{CF} \to \mathbb{S}(B\Gamma).$$

This is the map that 'forgets the control' over S^1. We will give a homological description of $\mathcal{CF}$.

We will study the homotopy colimit structure of the quotient map q_Γ. Equip S^1 with the structure of a simplicial complex with three 0-simplices v_i, $i = 0,1,3$ and three 1-simplices $e_i = \{v_i, v_{i+1}\}$, where i is taken mod 3. The projection map induces a functor:

$$\rho : \mathrm{cat}(S^1 \times B\mathcal{G}_\Gamma)^{op} \to \mathrm{cat}(S^1)^{op}.$$

Using Segal's Pushdown Construction we get a homotopy equivalence:

$$\mathrm{hocolim}_{\mathrm{cat}(S^1\times B\mathcal{G}_\Gamma)^{op}}(\mathbb{S}\circ \mathrm{bar}(q_\Gamma)) \simeq \mathrm{hocolim}_{\mathrm{cat}(S^1)^{op}}(P_*\rho).$$

We will describe explicitly the functor $P_*\rho$ on $\mathrm{cat}(S^1)^{op}$:

For each simplex t of S^1, let

$$\mathbb{S}\circ\text{bar}(q_\Gamma)(t,-) : \text{cat}(B\mathcal{G}_\Gamma)^{op} \to \mathbb{S}pectra,\ [\sigma]_\Gamma \mapsto \mathbb{S}\circ\text{bar}(q_\Gamma)(t,[\sigma]_\Gamma)$$

and thus

$$P_*\rho(t) = \text{hocolim}_{\text{cat}(B\mathcal{G}_\Gamma)^{op}}(\mathbb{S}\circ\text{bar}(q_\Gamma)(t,-)).$$

Now if we fix t, the functor $\text{bar}(q_\Gamma)$ associates to $[\sigma]_\Gamma$, the space

$$q_\Gamma^{-1}(\hat{t}\times\hat{\sigma}) = (EG\times\{\hat{t}\}\times\mathbb{Z})\times_\Gamma\Gamma\hat{\sigma} \cong (EG\times\{\hat{t}\}\times\mathbb{Z})\times_{\Gamma_\sigma}\hat{\sigma}.$$

The inclusion map $G \to \Gamma$ induces a commutative diagram:

$$\begin{array}{ccc} EG\times_G\mathcal{E}\mathcal{G}_\Gamma & \xrightarrow{\bar{u}} & (EG\times\{\hat{t}\}\times\mathbb{Z})\times_\Gamma\mathcal{E}\mathcal{G}_\Gamma \\ {\scriptstyle p_G}\downarrow & & \downarrow{\scriptstyle q_\Gamma} \\ \mathcal{E}\mathcal{G}_\Gamma/G = \mathcal{B}\mathcal{G}_G & \xrightarrow{u} & \hat{t}\times\mathcal{B}\mathcal{G}_\Gamma = \hat{t}\times\mathcal{E}\mathcal{G}_\Gamma/\Gamma \end{array}$$

where u is just the quotient map. Notice that $\mathcal{E}\mathcal{G}_\Gamma$ is also a model for the space of type $\mathcal{E}\mathcal{G}_G$. Then u induces a functor:

$$U : \text{cat}(\mathcal{B}\mathcal{G}_G)^{op} \to \text{cat}(\mathcal{B}\mathcal{G}_\Gamma)^{op}.$$

Start with the restriction:

$$p_G| : (p_G)^{-1}(|U{\downarrow}[\sigma]_\Gamma|) \to |U{\downarrow}[\sigma]_\Gamma|.$$

Define a functor

$$h : \text{cat}(\mathcal{B}\mathcal{G}_\Gamma)^{op} \to \mathbb{S}pectra,\ \ h([\sigma]_\Gamma) = \text{hocolim}_{\text{cat}(|U\downarrow[\sigma]_\Gamma|)^{op}}(\mathbb{S}\circ\text{bar}(p_G|)).$$

Claim. $h([\sigma]_\Gamma \simeq \mathbb{S}\circ\text{bar}(q_\Gamma)(t,[\sigma]_\Gamma)$ and the homotopy equivalence is natural in $[\sigma]_\Gamma$.

Proof. For the over category, we have:

$$U{\downarrow}[\sigma]_\Gamma = \{[\gamma\tau]_G : \tau \geq \sigma, \gamma \in \Gamma\} = \{[(1,m)\tau]_G : \tau \geq \sigma, m \in \mathbb{Z}\},$$

where $[-]_G$ denotes the G-orbit of the simplex. To describe a splitting of $p_G^{-1}(|U{\downarrow}[\sigma]_\Gamma|)$ we need to start with an equivalence relation on $\mathbb{Z}$. Every simplex σ of $\mathcal{E}\mathcal{G}_\Gamma$ defines an equivalence relation on $\mathbb{Z}$,

$$m_1{\sim_\sigma}m_2 \iff \Gamma_\sigma\cap(1,m_1)G(1,-m_2) \neq \emptyset,\ \text{ for each }\ m_1,m_2 \in \mathbb{Z}.$$

Also, for each $t \in S^1$, $m \in \mathbb{Z}$, we define a map

$$i_m : (EG\times\{t\}\times\{m\})\times_{\Gamma_\sigma\cap G}\hat{\sigma} \to (EG\times\{t\}\times\mathbb{Z})\times_\Gamma\Gamma\hat{\sigma},\ [(x,t,m),\hat{\sigma}] \mapsto [(x,t,m),\hat{\sigma}]$$

(here $EG{\times}\{t\}{\times}\mathbb{Z} \subset E\Gamma$). Notice that the image of i_m depends on the choice of $t \in S^1$. Direct calculations show that:

$$\begin{array}{lcl} |U{\downarrow}[\sigma]_\Gamma| & = & \coprod_{[m_k]} |\mathrm{Id}_{\mathrm{cat}(\mathcal{BG}_G)^{op}}{\downarrow}[(1,m_k)\sigma]_G| \\ p_G^{-1}(|U{\downarrow}[\sigma]_\Gamma|) & = & \coprod_{[m_k]} p_G^{-1}(|\mathrm{Id}_{\mathrm{cat}(\mathcal{BG}_G)^{op}}{\downarrow}[(1,m_k)\sigma]_G|) = \coprod_{[m_k]} EG{\times}_G G(|\mathrm{Id}_{\mathrm{cat}(\mathcal{E}G_\Gamma)^{op}}{\downarrow}(1,m_k)\sigma|) \\ & \cong & \coprod_{[m_k]} EG{\times}_{\Gamma_\sigma\cap G}|\mathrm{Id}_{\mathrm{cat}(\mathcal{E}G_\Gamma)^{op}}{\downarrow}(1,m_k)\sigma| \end{array}$$

where m_k runs over a complete set of representatives of $\sim_\sigma$. Thus

$$h([\sigma]_\Gamma) = \mathrm{hocolim}_{\mathrm{cat}(|U\downarrow[\sigma]_\Gamma|)^{op}}(\mathbb{S}\circ\mathrm{bar}(p_G|)) \simeq \bigvee_{[m_k]} \mathrm{hocolim}_{\mathrm{cat}(|\mathrm{Id}_{\mathcal{BG}_G}\downarrow[(1,m_k)\sigma]_G|)^{op}}(\mathbb{S}\circ\mathrm{bar}(p_G|))$$

But $[(1,m_k)\sigma]_G$ is a terminal object in the category $\mathrm{Id}_{\mathcal{BG}_G}{\downarrow}[(1,m_k)\sigma]_G$. Therefore,

$$h([\sigma_\Gamma) \simeq \bigvee_{[m_k]} \mathbb{S}(p_G^{-1}([(1,m_k)\hat{\sigma}]_G))$$

Continuing the calculations,

$$\begin{array}{lcl} h([\sigma]_\Gamma & \simeq & \mathbb{S}\left(\coprod_{[m_k]} p_G^{-1}([(1,m_k)\hat{\sigma}]_G)\right) = \mathbb{S}\left(\coprod_{[m_k]} EG{\times}_G G(1,m_k)\hat{\sigma}\right) \simeq \mathbb{S}\left(\coprod_{[m_k]} \mathrm{Im}(i_k)\right) \\ & \simeq & \mathbb{S}((EG{\times}\{\hat{t}\}{\times}\mathbb{Z}){\times}_\Gamma\Gamma\hat{\sigma}) = \mathbb{S}\circ\mathrm{bar}(q_\Gamma)(t,[\sigma]_\Gamma) \end{array}$$

The naturality follows from the construction.

Now Segal's Theorem applied to U implies:

$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_G)^{op}}(\mathbb{S}\circ\mathrm{bar}(p_G)) \simeq \mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_\Gamma)^{op}}(h) \simeq P_*\rho(t).$$

Thus, if we assume that G **satisfies the $\mathbb{S}$-IC**, for each $t \in \mathrm{cat}(S^1)^{op}$,

$$\mathbb{S}(BG) \simeq \mathrm{hocolim}_{\mathrm{cat}(\mathcal{BG}_G)^{op}}(\mathbb{S}\circ\mathrm{bar}(p_G)) \simeq P_*\rho(t)$$

Since

$$\mathrm{hocolim}_{\mathrm{cat}(S^1\times\mathcal{BG}_\Gamma)^{op}}(\mathbb{S}\circ\mathrm{bar}(q_\Gamma)) \simeq \mathrm{hocolim}_{\mathrm{cat}(S^1)^{op}}(P_*\rho) \simeq \mathrm{hocolim}_{\mathrm{cat}(S^1)^{op}}(BG),$$

the homological properties of homotopy colimits imply that (Section 3 in [70]) there is a long exact sequence

$$\cdots \to \pi_i(\mathbb{S}(BG)) \xrightarrow{1-\phi_*} \pi_i(\mathbb{S}(BG)) \to H_i(S^1{\times}B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma)) \to \pi_{i-1}(\mathbb{S}(BG)) \to \dots$$

Therefore we get that $\mathcal{CF} \simeq \mathbb{H}.(S^1{\times}B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma))$ Thus $\mathbb{H}.(S^1{\times}B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma))$ represents the controlled (over S^1) $\mathbb{S}$-theory.

The commutative diagram

$$\begin{array}{ccccc} E\Gamma\times_\Gamma\mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & E\Gamma\times_\Gamma\mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & B\Gamma \\ {\scriptstyle q_\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle p_\Gamma} & & \big\downarrow \\ S^1\times\mathcal{B}\mathcal{G}_\Gamma & \longrightarrow & \mathcal{B}\mathcal{G}_\Gamma & \longrightarrow & * \end{array}$$

induces a map of spectra $f : \mathbb{H}.(S^1\times B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma)) \to \mathbb{S}(B\Gamma)$. This map is the 'forget control' map. The inclusion induced map $\mathbb{S}(BG) \to \mathbb{S}(B\Gamma)$ factors through the spectrum $\mathbb{H}.(S^1\times B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma))$ because of the commutative diagram

$$\begin{array}{ccccccccc} BG & \longleftarrow & EG\times_G\mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & E\Gamma\times_\Gamma\mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & E\Gamma\times_\Gamma\mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & B\Gamma \\ \big\downarrow & & {\scriptstyle p_G}\big\downarrow & & {\scriptstyle q_\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle p_\Gamma} & & \big\downarrow \\ * & \longleftarrow & \mathcal{B}\mathcal{G}_G & \longrightarrow & S^1\times\mathcal{B}\mathcal{G}_\Gamma & \longrightarrow & \mathcal{B}\mathcal{G}_\Gamma & \longrightarrow & * \end{array}$$

Notice that the map in the second square is not natural because it depends on the choice of an element of S^1 but the composition induced by the third square is not affected by that choice. Thus the following diagram commutes, up to homotopy:

$$\begin{array}{ccc} \mathbb{S}(BG) & \longrightarrow & \mathbb{H}.(S^1\times B\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma)) \\ {\scriptstyle \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \mathbb{S}(BG) & \longrightarrow & \mathbb{S}(B\Gamma) \end{array}$$

For the Bundle version, start with a commutative diagram:

$$\begin{array}{ccccccc} \bar{\bar{Y}} & \longrightarrow & \bar{\bar{Y}} & \longrightarrow & \bar{Y} & \longrightarrow & Y \\ {\scriptstyle \bar{\bar{\rho}}}\big\downarrow & & {\scriptstyle \bar{\bar{\rho}}}\big\downarrow & & {\scriptstyle \bar{\rho}}\big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ EG\times_G\mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & EG\times_G\mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & E\Gamma\times_\Gamma\mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & B\Gamma \\ {\scriptstyle q_G}\big\downarrow & & {\scriptstyle p_G}\big\downarrow & & {\scriptstyle p_\Gamma}\big\downarrow & & \big\downarrow \\ S^1\times\mathcal{B}\mathcal{G}_G & \longrightarrow & \mathcal{B}\mathcal{G}_G & \longrightarrow & \mathcal{B}\mathcal{G}_\Gamma & \longrightarrow & * \end{array}$$

where ρ is a bundle and the top diagrams are pull-back diagrams. As before, we have the following.

Theorem 4.7. *Assume that the Bundle $\mathbb{S}$-IC holds for G. Then there is an exact sequence:*

$$\cdots \to \pi_i(\mathbb{S}(\bar{\bar{Y}})) \xrightarrow{1-\phi_*} \pi_i(\mathbb{S}(\bar{\bar{Y}})) \to H_i(S^1\times B\mathcal{G}_\Gamma, \mathbb{S}(\bar{\rho}\circ q_\Gamma)) \to \pi_{i-1}(\mathbb{S}(\bar{\bar{Y}})) \to \dots$$

where $\phi : \bar{\bar{Y}} \to \bar{\bar{Y}}$ is the homeomorphism induced by α.

Remark 4.8. The groups $H_*(S^1\times B\mathcal{G}_\Gamma, \mathbb{S}(\bar{\rho}\circ q_\Gamma))$ are the analogues of the controlled groups over S^1. Theorem 4.7 states the homological properties of the controlled groups.

We specialize to the case $\mathbb{S} = \mathbb{K}_R$, the K-theory spectrum with coefficients in a ring R. Then there is a commutative diagram of exact sequences for each i:

$$\begin{array}{ccccccc} & & & H_i(S^1 \times \mathcal{B}\mathcal{G}_\Gamma, \mathbb{K}_R(q_\Gamma)) & & & \\ & & \nearrow & & \searrow & & \\ K_i(RG) & \xrightarrow{1-\phi_*} K_i(RG) & & \Big\downarrow f & & K_{i-1}(RG) & \xrightarrow{1-\phi_*} K_{i-1}(RG) \\ & & \searrow & & \nearrow & & \\ & & & K_i(R\Gamma) & & & \end{array}$$

If the Nil-groups of RG vanish, (for example, if RG is regular coherent ring), the bottom sequence is exact for the Bass–Heller–Swan formula ([89], [90]). Thus, in this case,

$$f : H_i(S^1 \times \mathcal{B}\mathcal{G}_\Gamma, \mathbb{K}_R(q_\Gamma)) \to K_i(R\Gamma)$$

is an isomorphism for all $i \in \mathbb{Z}$. From the definition of f we have that it factors as:

$$f : H_i(S^1 \times \mathcal{B}\mathcal{G}_\Gamma, \mathbb{K}_R(q_\Gamma)) \xrightarrow{p} H_i(\mathcal{B}\mathcal{G}_\Gamma, \mathbb{K}_R(p_\Gamma)) \xrightarrow{A} K_i(R\Gamma)$$

where p is induced by the projection to the second coordinate and A is the assembly map that appears in the Isomorphism Conjecture.

Theorem 4.9. *Assume that the Nil-groups of RG vanish. Then*

$$A : H_i(\mathcal{B}\mathcal{G}_\Gamma, \mathbb{K}_R(p_\Gamma)) \to K_i(R\Gamma)$$

is an epimorphism.

4.2. **Controlled Groups over the Interval.** Let $\Gamma = G_1 *_{G_0} G_2$, where G_0 is a subgroup of $G_1 \cap G_2$. Let BG_i, $i = 0, 1, 2$, be classifying spaces for the corresponding groups with BG_0 a subcomplex of $BG_1 \cap BG_2$. Choose $B\Gamma$ to be the double mapping cylinder of the inclusion maps. Then there is a natural map $\rho : B\Gamma \to I$, where I is the unit interval. Let $E\Gamma$ be the universal cover of $B\Gamma$. Let

$$E\Gamma \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma \xrightarrow{q_\Gamma} I \times \mathcal{B}\mathcal{G}_\Gamma$$

be maps induced by the natural projection. As before, we have the commutative diagram

$$\begin{array}{ccc} E\Gamma \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma & \longrightarrow & E\Gamma \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma \\ {\scriptstyle q_\Gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle p_\Gamma} \\ I \times \mathcal{B}\mathcal{G}_\Gamma & \longrightarrow & \mathcal{B}\mathcal{G}_\Gamma \end{array}$$

We will show that $\mathbb{H}.(I \times \mathcal{B}\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma))$ satisfies a Mayer–Vietoris type property.

We work as in Section 4.1. Equip I with the structure of a simplicial complex with one 1-simplex I, and two 0-simplices $0, 1$. The projection map induces a functor:

$$\rho : \mathrm{cat}(I \times B\mathcal{G}_\Gamma)^{op} \to \mathrm{cat}(I)^{op}$$

We will use Segal's Pushdown Construction. We start by giving a description of the functor $P_*\rho$ on $\text{cat}(I)^{op}$. For each simplex t of I, define a functor:

$$\mathbb{S}\circ\text{bar}(q_\Gamma)(t,-) : \text{cat}(B\mathcal{G}_\Gamma)^{op} \to \mathbb{S}pectra,\ [\sigma]_\Gamma \mapsto \mathbb{S}\circ\text{bar}(q_\Gamma)(t,[\sigma]_\Gamma)$$

Then the functor $P_*\rho$ is defined as:

$$P_*\rho : \text{cat}(I)^{op} \to \mathbb{S}pectra,\ P_*\rho(t) = \text{hocolim}_{\text{cat}(B\mathcal{G}_\Gamma)^{op}}(\mathbb{S}\circ\text{bar}(q_\Gamma)).$$

So if we fix t, the functor $\text{bar}(q_\Gamma)$ associates to $[\sigma]_\Gamma$, the space

$$q_\Gamma^{-1}(\hat{t}\times\hat{\sigma}) = (EG_i)\Gamma\times_\Gamma\Gamma\hat{\sigma} \cong (EG_i)\Gamma\times_{\Gamma_\sigma}\hat{\sigma},$$

after subdividing, where $i=0$ if t is an interior point and $G_i = G_{t+1}$ when $t \in \partial I$. Actually, the inverse image for $t \in \text{Int}(I)$, is

$$q_\Gamma^{-1}(\hat{t}\times\hat{\sigma}) = (EG_i\times\{t\})\Gamma\times_\Gamma\Gamma\hat{\sigma}$$

The inclusion map $G_i \to \Gamma$ induces a commutative diagram:

$$\begin{array}{ccc} EG_i\times_{G_i}\mathcal{E}\mathcal{G}_\Gamma & \xrightarrow{\bar{u}} & EG_i\Gamma\times_\Gamma\mathcal{E}\mathcal{G}_\Gamma \\ {\scriptstyle p_{G_i}}\big\downarrow & & \big\downarrow{\scriptstyle q_\Gamma} \\ \mathcal{E}\mathcal{G}_\Gamma/G_i = \mathcal{B}\mathcal{G}_{G_i} & \xrightarrow{u} & \hat{t}\times\mathcal{B}\mathcal{G}_\Gamma = \hat{t}\times\mathcal{E}\mathcal{G}_\Gamma/\Gamma \end{array}$$

where u is just the quotient map. As in the last section, u induces a functor

$$U : \text{cat}(\mathcal{B}\mathcal{G}_{G_i})^{op} \to \text{cat}(\mathcal{B}\mathcal{G}_\Gamma)^{op}.$$

We consider the restriction

$$p_{G_i}| : (p_{G_i})^{-1}(|U{\downarrow}[\sigma]_\Gamma|) \to |U{\downarrow}[\sigma]_\Gamma|$$

and define a functor

$$h : \text{cat}(\mathcal{B}\mathcal{G}_\Gamma)^{op} \to \mathbb{S}pectra,,\ \ h([\sigma]_\Gamma) = \text{hocolim}_{\text{cat}(|U\downarrow[\sigma]_\Gamma|)^{op}}(\mathbb{S}\circ\text{bar}(p_{G_i}|)).$$

As in the last section, we get that there is a natural homotopy equivalence:

$$h([\sigma_\Gamma]) \simeq \mathbb{S}\circ\text{bar}(q_\Gamma)(t,[\sigma_\Gamma]).$$

We will indicate the modifications we need to get the result. Choose coset representatives:

$$\Delta_i = \{\gamma_{i,j} :\ \gamma_{i,j} \in \mathcal{A}_i\},\ i = 0,1,2.$$

In other words,

$$\Gamma = \coprod_{j\in\mathcal{A}_i} G_i\gamma_{i,j},\ i = 0,1,2.$$

With the above notation:

- Every simplex $[\sigma]_\Gamma$ of $\mathcal{E}\mathcal{G}_\Gamma$ defines an equivalence relation on Δ_i,

$$\gamma_{i,j}\sim_\sigma\gamma'_{i,j} \iff \Gamma_\sigma\cap\gamma_{i,j}G_i(\gamma'_{i,j})^{-1} \neq \emptyset.$$

- Let $t \in I$. Set $i = 0$ if $t \in \mathrm{Int}(I)$ and $i = t$ if $t \in \partial I$. For each $\gamma_{i,j} \in \Delta_i$, a map
$$\iota_{\gamma_{i,j}} : EG_i \times_{\Gamma_\sigma \cap G_i} g_{i,j}\hat{\sigma} \to EG_i\Gamma \times_\Gamma \Gamma\hat{\sigma},\ [x, \gamma_{i,j}\hat{\sigma}] \mapsto [x, \gamma_{i,j}\hat{\sigma}]$$
here $EG_i\Gamma \subset E\Gamma$. Notice that the image of $\iota_{\gamma_{i,j}}$ depends on the choice of $t \in I$.
- There is a homeomorphism:
$$\chi_i : EG_i \times_{G_i} G_i\gamma_{i,j}\hat{\sigma} \to E\Gamma \times_\Gamma \mathcal{E}\mathcal{G}_\Gamma$$
onto $\mathrm{Im}(\iota_{\gamma_{i,j}})$.

Then
$$\begin{aligned} |U{\downarrow}[\sigma_\Gamma]| &= \coprod_{[} \gamma_{i,j}]|\mathrm{Id}_{\mathrm{cat}(\mathcal{B}\mathcal{G}_{G_i})^{op}}{\downarrow}[\gamma_{i,j}\sigma]_{G_i}| \\ (p_{G_i})^{-1}(|U{\downarrow}[\sigma_\Gamma]|) &= \coprod_{[} \gamma_{i,j}]EG_i \times_{\Gamma_\sigma \cap G_i} |\mathrm{Id}_{\mathrm{cat}(\mathcal{B}\mathcal{G}_\Gamma)^{op}}{\downarrow}\gamma_{i,j}\sigma|. \end{aligned}$$

Segal's Theorem implies that
$$\mathrm{hocolim}_{\mathrm{cat}(\mathcal{B}\mathcal{G}_{G_i})^{op}}(\mathbb{S}\circ\mathrm{bar}(p_{G_i})) \simeq \mathrm{hocolim}_{\mathrm{cat}(\mathcal{B}\mathcal{G}_\Gamma)^{op}}(h) \simeq P_*\rho(t).$$

Thus, if the $\mathbb{S}$-IC holds for G_i, $i = 0, 1, 2$, there is a natural homotopy equivalence:
$$P_*\rho(t) = \mathrm{hocolim}_{\mathrm{cat}(B\mathcal{G}_\Gamma)^{op}}(\mathbb{S}\circ\mathrm{bar}(q_\Gamma)(t, -)) \simeq \begin{cases} \mathbb{S}(BG_0), & \text{if } t \in \mathrm{Int}(I) \\ \mathbb{S}(BG_{t+1}), & \text{if } t \in \partial I. \end{cases}$$

Again, using the homological properties of homotopy colimits, we can show that the following is a homotopy cartesian diagram:
$$\begin{array}{ccc} \mathbb{S}(BG_0) & \longrightarrow & \mathbb{S}(BG_1) \\ \big\downarrow & & \big\downarrow \\ \mathbb{S}(BG_1) & \longrightarrow & \mathbb{H}.(I \times \mathcal{B}\mathcal{G}_\Gamma, \mathbb{S}(q_\Gamma)) \end{array}$$

In analogy with the circle, for the case $\mathbb{S} = \mathbb{K}_R$ we have that:

Theorem 4.10. *Let R be a ring such that the Waldhausen Nil-groups of RG_0 vanish. Then*

(1) *the forgetful map*
$$f : \mathbb{H}.(I \times \mathcal{B}\mathcal{G}_\Gamma : \mathbb{K}_R(q_\Gamma)) \to \mathbb{K}_R(B\Gamma)$$
is a homotopy equivalence. This map is induced by the universal properties of the homotopy cartesian square.

(2) *the assembly map*
$$A : \mathbb{H}.(\mathcal{B}\mathcal{G}_\Gamma, \mathbb{K}_R(p_\Gamma)) \to \mathbb{K}_R(B\Gamma)$$
induces an epimorphism on homotopy.

5. Applications

We will apply the results to prove that $\mathbb{K}_R$-IC is true for certain types of groups. Most of the results in this section were obtained, with different methods, in [10]. Before we start we state the Novikov Conjecture for a discrete group Γ. Let $\mathcal{C}_\Gamma$ be the classifying space for the class of finite subgroups of Γ. Again, there is a commutative diagram:

$$\begin{array}{ccc} E\Gamma\times_\Gamma\mathcal{E}\mathcal{C}_\Gamma & \longrightarrow & B\Gamma \\ {\scriptstyle p}\big\downarrow & & \big\downarrow \\ \mathcal{B}\mathcal{C}_\Gamma & \longrightarrow & * \end{array}$$

A group Γ satisfies the **integral $\mathbb{S}$-Novikov Conjecture** if the assembly map

$$A_{\mathcal{C}} : \mathbb{H}.(\mathcal{B}\mathcal{C}_\Gamma, \mathbb{S}(p)) \to \mathbb{S}(B\Gamma)$$

induces a split injection on the homotopy groups. It is an open question if all the torsion free groups satisfy the integral $\mathbb{K}_R$-Novikov Conjecture. It was proved that groups of finite cohomological dimension that also have finite asymptotic dimension satisfy the integral $\mathbb{K}_R$-Novikov Conjecture ([8]), generalizing the calculations in [18].

Remark 5.1. The geometric methods used in proving the Novikov Conjecture depend on controlling how 'large' compact subsets of $E\Gamma$ become as they are translated to infinity through the Γ-action. The assumption on the asymptotic dimension guarantees such control.

Let R be a regular Noetherian ring. Let G be a torsion free group that satisfies the integral $\mathbb{K}_R$-Novikov Conjecture. The commutative diagram

$$\begin{array}{ccccc} EG\times_G\mathcal{E}\mathcal{C}_G & \longrightarrow & EG\times_G\mathcal{E}\mathcal{G}_G & \longrightarrow & BG \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p_G} & & \big\downarrow \\ \mathcal{B}\mathcal{C}_G & \longrightarrow & \mathcal{B}\mathcal{G}_G & \longrightarrow & * \end{array}$$

provides a splitting of the assembly map $A_{\mathcal{C}} = A\circ A_{\mathcal{C},\mathcal{G}}$:

$$\begin{array}{ccc} \mathbb{H}.(\mathcal{B}\mathcal{C}_G, \mathbb{K}_R(p)) & \xrightarrow{A_{\mathcal{C}}} & \mathbb{K}_R(BG) \\ {\scriptstyle A_{\mathcal{C},\mathcal{G}}}\big\downarrow & \nearrow{\scriptstyle A} & \\ \mathbb{H}.(\mathcal{B}\mathcal{G}_G, \mathbb{K}_R(p_G)) & & \end{array}$$

The relative assembly map is the map induced by the first commutative square. For every virtually cyclic group S of G, let $\mathcal{C}_S$ be the class of finite subgroups of S. Theorem A.10 in [43] states that $A_{\mathcal{C},\mathcal{G}}$ is an equivalence if the assembly map $A_{\mathcal{C}_S}$ is an equivalence for all virtually cyclic subgroups S of G (a similar calculation appears in [31]). Since G is torsion free, the only finite subgroup of G is the trivial group and the only virtually cyclic subgroups are infinite cyclic subgroups. Thus $A_{\mathcal{C}_S}$ is the assembly map in the integral $\mathbb{K}_R$-Novikov conjecture for $S \cong \mathbb{Z}$. Since R is regular Noetherian, this map is an equivalence ([43], Remark A.11). Therefore, if $A_{\mathcal{C}}$ induces a monomorphism on homotopy groups, so does A. Combining with the results in Theorems 4.9 and 4.10, we get the following:

Theorem 5.2. *Let G be a torsion free group and R a regular Noetherian ring. Assume that G satisfies the $\mathbb{K}_R$-IC, and that RG is a regular Noetherian ring. Let Γ be a torsion free group defined by:*

(i) $\Gamma = G \rtimes \mathbb{Z}$, *or*
(ii) $\Gamma = G_1 *_G G_2$, *such that G_i, $i = 1, 2$, satisfy the $\mathbb{K}_R$-IC,*

such that Γ satisfies the integral $\mathbb{K}_R$-Novikov conjecture. Then Γ satisfies the $\mathbb{K}_R$-IC.

Remark 5.3. We assume that the groups involved satisfy the $\mathbb{K}_R$-IC as in the Theorem 5.2. More generally, it is enough if we assume that R is regular Noetherian ring, and the twisted Nil-groups of RG vanish in (i) and the Waldhausen Nil-groups of the triple $(RG, R[G_1\backslash G], R[G_2\backslash G])$ vanish in (ii).

Now we give applications of the Theorem.

Corollary 5.4. *Let G be a torsion free group and R a regular Noetherian ring. Assume that:*

(1) *G satisfies the $\mathbb{K}_R$-IC,*
(2) *G has finite asymptotic dimension,*
(3) *RG is a regular coherent ring.*

Let Γ be a torsion free group defined by:

(i) $\Gamma = G \rtimes \mathbb{Z}$, *or*
(ii) $\Gamma = G_1 *_G G_2$, *such that G_i, $i = 1, 2$, satisfy the $\mathbb{K}_R$-IC and they have finite cohomological and asymptotic dimensions.*

Then Γ satisfies the $\mathbb{K}_R$-IC.

Proof. The assumptions on the groups imply that Γ has finite asymptotic dimension ([13]). Also, Γ has finite cohomological dimension and thus it admits a finite dimensional $B\Gamma$. By [8], Γ satisfies the integral $\mathbb{K}_R$-Novikov conjecture. The result follows from Theorem 5.2. □

Corollary 5.5. *Let F be a finitely generated free group and R a regular Noetherian ring. Then*

(1) *F satisfies the $\mathbb{K}_R$-IC.*
(2) *$F \rtimes \mathbb{Z}$ satisfies the $\mathbb{K}_R$-IC*

Proof. The first statement follows by induction on the number k of generators of F: If $k = 2$, then $F = \mathbb{Z} * \mathbb{Z}$ and the result follows from 5.4. For $k > 2$, $F = F_{k-1} * \mathbb{Z}$, where F_{k-1} is the free group on $(k - 1)$ generators. All the assumptions of Corollary 5.4 are satisfied and thus F satisfies the $\mathbb{K}_R$-IC.

For the second statement, we use again Corollary 5.4. Part (1) implies that assumption (1) is satisfied. Also, the free group has finite asymptotic dimension and thus assumption (2) is satisfied. Assumption (3) follows because R is regular Noetherian ring. □

Remark 5.6. In [4] and [46], there was a special assumption for groups of type $F \rtimes \mathbb{Z}$ to satisfy the fibered pseudoisotopy IC. Essentially the assumption was that the action of $\mathbb{Z}$ on F has certain geometric properties. Corollary 5.5 is more general because such an assumption is not needed but it only gives the $\mathbb{K}_R$-IC and not the fibered version. A general fibered version could not follow along the same lines because the assumptions in Corollary 5.4 guarantee that all the Nil-groups that appear vanish.

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