MANIFESTATIONS OF NON LINEAR ROUNDNESS IN ANALYSIS, DISCRETE GEOMETRY AND TOPOLOGY

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Abstract. Some forty years ago Per Enflo introduced the nonlinear notions of roundness and generalized roundness for general metric spaces in order to study (a) uniform homeomorphisms between (quasi-) Banach spaces, and (b) Hilbert's *Fifth Problem* in the context of non locally compact topological groups (see [23], [24], [25], and [26]). Since then the concepts of roundness and generalized roundess have proven to be particularly useful and durable across a number of important mathematical fields such as coarse geometry, discrete geometry, functional analysis and topology. The purpose of this article is to take a retrospective look at some notable applications of versions of *nonlinear roundness* across such fields, to draw some hitherto unpublished connections between such results, and to highlight some very intriguing open problems.

1. Nonlinear Roundness — Introduction and Background

Nonlinear notions of *roundness* and *generalized roundness* (see Definition 1.1) were introduced by Enflo in the late 1960s in a series of concise but elegant papers [23], [24], [25] and [26]. The purpose of Enflo's programme of study in these papers was to investigate Hilbert's fifth problem in the context of non locally compact topological groups and to address the nonlinear classification of topological vector spaces up to uniform homeomorphism. Therein, Enflo used both roundness and generalized roundness in order to expose decisive estimates on the distortion of certain nonlinear maps between metric spaces. Later, within the context of Banach spaces, it became clear that roundness could be viewed as a natural precursor of Rademacher type. On the other hand, kernels of negative type and generalized roundness are known to be very tightly related. Issues of distortion and type now figure prominently in several modern fields of mathematical research and (as a result) ideas surrounding roundness and generalized roundness continue to find new and sometimes unexpected areas of applicability. Our purpose in this article is to present and unify some of the recent developments along these lines, and to present a number of open problems and avenues for further research.

The following definitions are central to the entire paper and are therefore collected together here for easy reference throughout the subsequent sections.

Definition 1.1. Let $p \geq 0$ and let (X,d) be a metric space. Then:

(a) (X,d) has *roundness* p if and only if for all quadruples $x_{00}, x_{01}, x_{11}, x_{10} \in X$, we have:

$$\begin{aligned} d(x_{00},x_{11})^p + d(x_{01},x_{10})^p \quad &\leq \quad d(x_{00},x_{01})^p + d(x_{01},x_{11})^p + \\ & \quad\quad d(x_{11},x_{10})^p + d(x_{10},x_{00})^p. \end{aligned}$$

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(b) (X,d) has *generalized roundness* p if and only if for all natural numbers $n \in \mathbb{N}$, and all choices of points $a_1, \ldots, a_n, b_1, \ldots, b_n \in X$, we have:

$$\sum_{1 \leq k < l \leq n} \left\{ d(a_k, a_l)^p + d(b_k, b_l)^p \right\} \quad \leq \quad \sum_{1 \leq j, i \leq n} d(a_j, b_i)^p. \tag{1}$$

(c) (X,d) has p*-negative type* if and only if for all natural numbers $n \geq 2$, all finite subsets $\{x_1, \ldots, x_n\} \subseteq X$, and all choices of real numbers $\eta_1, \ldots, \eta_n$ with $\eta_1 + \cdots + \eta_n = 0$, we have:

$$\sum_{1 \leq i,j \leq n} d(x_i, x_j)^p \eta_i \eta_j \quad \leq \quad 0.$$

(d) (X,d) has *strict* p*-negative type* if and only if it has p-negative type and the inequality in (c) is strict whenever the scalar n-tuple $(\eta_1, \ldots, \eta_n) \neq \vec{0}$.

Remark 1.2. In making Definition 1.1 (b) it is important to point out that repetitions among the a's and b's are allowed. Indeed, allowing such repetitions is essential to the general theory. We may, however, when making Definition 1.1 (b), assume that $a_j \neq b_i$ for all i, j $(1 \leq i, j \leq n)$. This is due to an elementary cancellation of like terms phenomenon that was first observed by Andrew Tonge (unpublished).

Clearly, roundness is just what one obtains by restricting Definition 1.1 (b) to the particular natural number $n = 2$. It is easy to see that every metric space has roundness one and generalized roundness zero. A midpoint convex (that is, metrically convex) metric space cannot have roundness p for any $p > 2$. Hilbert spaces have roundness 2 and generalized roundness 2 due to easy adaptations of the parallelogram law. The complete bipartite graphs $K_{n,n}$ can be metrized to provide examples of finite metric spaces whose maximal generalized roundness decreases to zero as n tends to infinity.[1] This is noted in the introduction to Enflo [24] and illustrates that roundness and generalized roundness are distinct notions.

In some situations it is natural to consider the supremum of all p for which a given metric space has (generalized) roundness p.

Definition 1.3. Let (X,d) be a metric space. Then:

(a) $\mathfrak{p}_X = \sup\{p : (X,d) \text{ has roundness } p\}$ is called the *maximal roundness* of the metric space (X,d), and

(b) $\mathfrak{q}_X = \sup\{p : (X,d) \text{ has generalized roundness } p\}$ is called the *maximal generalized roundness* of the metric space (X,d).

The terminology "maximal" in Definition 1.3 is easily justified. A simple argument shows that the set of all p for which a given metric space (X,d) has roundness p is a closed subset of $[0, \infty)$. The same goes for generalized roundness. Hence (X,d) has roundness $\mathfrak{p}_X$ and generalized roundness $\mathfrak{q}_X$. In other words, metric spaces always attain their supremal (generalized) roundness. Typically, given a metric

[1] Indeed, simply represent $K_{n,n}$ as distinct vertices $a_1, \ldots, a_n, b_1, \ldots, b_n$ and define a metric on $K_{n,n}$ via $d(a_i, a_j) = 2 = d(b_i, b_j)$ for all $1 \leq i < j \leq n$, and $d(a_i, b_j) = 1$ for all $1 \leq i, j \leq n$. Then $(K_{n,n}, d)$ does not have generalized roundness p for any $p > -\log_2\big(1 - \frac{1}{n}\big)$. On the other hand—and at the other extreme—if we define all edges in $K_{n,n}$ to have length 1 (in other words, if we consider the ordinary graph metric on $K_{n,n}$), then we obtain a metric space that has maximal roundness ∞. (This is, of course, true for any set endowed with the discrete metric.)

space (X, d), computing $\mathfrak{p}_X$ and $\mathfrak{q}_X$, or even computing meaningful lower bounds on $\mathfrak{p}_X$ and $\mathfrak{q}_X$, is a difficult nonlinear problem. A major point of this paper is that (maximal) roundness and (maximal) generalized roundness can provide examples of coarse, uniform and isometric invariants when cast in the appropriate light. Some more preliminaries are in order before we can begin to expose such applications of *nonlinear roundness*[2].

It is the case that Definition 1.1 (a) can be rephrased in terms of nonlinear two-dimensional cubes and that, moreover, higher dimensional analogues of the roundness inequalities necessarily hold by induction. This turns out to be a crucial feature of roundness and the basic ideas are as follows.

Definition 1.4. Let n be a natural number. By an *n-cube* in a metric space (X, d) we simply mean the encoded range $N = \{x_\varepsilon\}$ of a function $f : \{0, 1\}^n \to X : \varepsilon \mapsto x_\varepsilon$ whose domain is the standard n-dimensional cube of all n-vectors $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ with coordinates chosen from the set $\{0, 1\}$.

An unordered pair of vertices $(x_\varepsilon, x_\delta)$ in an n-cube N is called a *diagonal* if $\varepsilon_i \neq \delta_i$ for all $i \in \{1, 2, \ldots, n\}$, and an *edge* if $\varepsilon_i \neq \delta_i$ for precisely one $i \in \{1, 2, \ldots, n\}$.

Notation. Given an n-cube $N = \{x_\varepsilon\}$ in a metric space (X, d), the set of all diagonals in N will be denoted $D(N)$, and the set of all edges in N will be denoted $E(N)$. Clearly $|D(N)| = 2^{n-1}$, and $|E(N)| = n \cdot 2^{n-1}$. Moreover, for any unordered pair of vertices $f = (x_\varepsilon, x_\delta)$ in N we will use $l(f)$ as a shorthand for the metric length $d(x_\varepsilon, x_\delta)$. This allows for an efficient method of writing down roundness related inequalities. For example, we can restate the condition in Definition 1.1 (a) as a statement about all 2-cubes $N = \{x_{ij}\}$ in X:

$$\sum_{d \in D(N)} l(d)^p \leq \sum_{e \in E(N)} l(e)^p.$$

Enflo [23] showed that roundness has the following extremely useful inductive property.

Theorem 1.5 (Enflo [23])**.** *If N is an n-cube in a metric space (X, d) that has roundness p, then we have:*

$$\sum_{d \in D(N)} l(d)^p \leq \sum_{e \in E(N)} l(e)^p.$$

In particular; if $d_{\min}$ denotes a diagonal of minimal d-length in N and $e_{\max}$ denotes an edge of maximal d-length in N, then we must have $l(d_{\min}) \leq n^{\frac{1}{p}} \cdot l(e_{\max})$.

The inequalities of Theorem 1.5 are particularly decisive in nonlinear settings if it is the case that $p > 1$. Indeed, Enflo [23] computed that $L_p(\mu)$—where $1 \leq p \leq 2$ and μ is a positive Borel measure on some sigma algebra (Ω, Σ)—has maximal roundness p, and then went on to obtain the following two results as consequences of Theorem 1.5.

Theorem 1.6 (Enflo [23])**.** *Let $1 \leq p \leq 2$ and let (X, d) be a metric space with roundness $q > p$. If $T : L_p(\mu) \to (X, d)$ is a uniform homeomorphism, then T^{-1} cannot satisfy a Lipschitz condition of order α at large distances for any $\alpha < \frac{q}{p}$. And so, $L_{p_1}(\mu_1)$ and $L_{p_2}(\mu_2)$ are not uniformly homeomorphic if $1 \leq p_1 < p_2 \leq 2$.*

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[2]We use this term as an umbrella for number of nonlinear versions of "type"; including roundness, generalized roundness, scaled Enflo type, Markov type, and so on

The proof of Theorem 1.6 determines (as an auxillary application) that any embedding of the *Hamming Cube* $\left(\{0,1\}^n, d_{\ell_1}\right)$ into Hilbert space must incur a distortion of at least $\sqrt{n}$. This was likely the first result to show an unbounded distortion for embeddings into Hilbert space. See, for example, Matousek [47] (Theorem 15.4.1).

Theorem 1.7 (Enflo [25])**.** *A non normable quasi-Banach space cannot be uniformly homeomorphic to any Banach space that has roundness greater than one. And so, $L_{p_1}(\mu_1)$ is not uniformly homeomorphic to $L_{p_2}(\mu_2)$ if $0 < p_1 < 1 < p_2 < \infty$.*

In Sections 4 and 5 of this paper we will indicate some generalizations of Theorems 1.6 and 1.7 which are obtained by replacing the notion of roundness with something more relaxed; namely, metric (or, BMW) type.

Prior to a preliminary discussion of generalized roundness and negative type we recall the following notion from the uniform theory of Banach spaces.

Definition 1.8. A metric space (X, d) is called a *universal uniform embedding space* if for each separable metric space (Y, ρ) there exists some subset $Z = Z(X) \subseteq X$ such that (Y, ρ) is uniformly homeomorphic to (Z, d).

Enflo [24] introduced generalized roundness as a means to address *Smirnov's Question:* Is every separable metric space uniformly homoemorphic to a subset of $L_2[0,1]$? This is a delicate question because every separable metric space is isometric to a subset of $C[0,1]$ and all separable Banach spaces are mutually homeomorphic by a remarkable theorem of Kadec [38]. Hence every separable metric space is homeomorphic to a subset of $L_2[0,1]$. Enflo [24] gave a negative answer to Smirnov's Question by showing that Hilbert spaces have generalized roundness two whereas no universal uniform embedding space can have positive generalized roundness. More recently, the ideas and constructions in Enflo [24] have had significant applications in the study of coarse geometry, particularly in relation to Gromov's [29] concept of coarse embeddings. We shall return to this important point and discuss it more scrupulously in Section 3.

The p-negative type inequalities of Definition 1.1 (c) arose in the 1920s and 1930s in connection with the study of isometric embeddings of metric spaces. In particular, Schoenberg [62], [63] was the first to use notions of positive and negative definite kernels to intensively study such embeddings. For example, Schoenberg [62] obtained that a metric space (X, d) admits an isometric embedding into a Hilbert space if and only if it has 2-negative type. Much later, Bretagnolle *et al* [13] obtained the following celebrated characterization of those real (quasi) normed linear spaces which are linearly isometric to a subspace of some $L_p(\mu)$-space, $0 < p \leq 2$.

Theorem 1.9 (Bretagnolle, Dacunha-Castelle and Krivine [13])**.** *Let $0 < p \leq 2$. Let $(X, \|\cdot\|)$ be a real quasi-normed space. Then: X is linearly isometric to a subspace of some $L_p(\mu)$-space if and only if X has p-negative type.*

There are versions of Theorem 1.9 that deal with the less tractable (commutative) case $p > 2$. We refer the reader to Koldobsky and König [42] for a discussion of results along these lines. There are also noncommutative versions of Theorem 1.9. For example, Junge [36], and (subsequently) Junge and Parcet [37], have obtained fundamental results on operator space embeddings of (noncommutative) L_q-spaces into L_p-spaces, $1 \leq p < q \leq 2$.

Interestingly, while one might hope for a purely metric version of Theorem 1.9 in the case $p < 2$, Lee and Naor [44] have given examples to show this is not possible. For $1 \leq p < 2$, Lee and Naor construct metric spaces that have p-negative type but which do no embed isometrically, or even bi-Lipschitzly, into any $L_p(\mu)$-space. Their work was motivated in part by (a desire to find a simpler example than) Khot and Vishnoi's [41] celebrated counter-example to the *Goeman's-Linial conjecture*: Every metric space of 1-negative type embeds with $O(1)$ distortion into L_1. The results of this paragraph are thus tightly related to the *Sparsest Cut problem with general demands*, a famous open problem in the study of approximation algorithms.

Lennard *et al* [45] have shown that generalized roundness p and p-negative type are equivalent notions in the category of metric spaces.

Theorem 1.10 (Lennard, Tonge and Weston [45])**.** *Let $p \geq 0$. For a metric space (X, d), the following are equivalent:*

(a) *(X, d) has p-negative type.*
(b) *(X, d) has generalized roundness p.*

Proof. The argument that (a) implies (b) is quite short and illuminating so we shall give it here. Suppose that (X, d) has p-negative type. Let $n \in \mathbb{N}$ be given. Consider points $a_1, \ldots, a_n, b_1, \ldots, b_n \in X$. For each k, $1 \leq k \leq n$, set $x_{2k-1} = a_k$ and $x_{2k} = b_k$. And for each j, $1 \leq j \leq 2n$, set $\eta_j = (-1)^j$. As $\sum_{j=1}^{2n} \eta_j = 0$ it follows from our hypothesis on (X, d) that we have:

$$\sum_{1 \leq i,j \leq 2n} d(x_i, x_j)^p \eta_i \eta_j \quad \leq \quad 0. \tag{2}$$

Summing over (i, j) both odd, (i, j) both even, i even and j odd, and i odd and j even, we see from (2) that:

$$\begin{aligned} 0 \quad \geq & \sum_{1 \leq i,j \leq n} \Big\{ d(a_i, a_j)^p + d(b_i, b_j)^p - 2d(a_i, b_j)^p \Big\} \\ = & \sum_{1 \leq i,j \leq 2n} d(x_i, x_j)^p \eta_i \eta_j. \end{aligned} \tag{3}$$

Hence (X, d) has generalized roundness p. Moreover, notice that if the inequality in (2) is strict, then the corresponding generalized roundness p inequality (3) must be strict too. (This illustrates that strict p-negative type implies "strict" generalized roundness p. The converse is also true. See Theorem 6.8 in Section 6.)

The argument that (b) implies (a) relies on the introduction of weighted versions of the generalized roundness inequalities. See Theorem 6.5 and Remark 6.6 in Section 6 for more details. □

Via Theorem 1.10 Lennard *et al* [45] were able to codify a few basic properties of generalized roundness that had not previously come to light. In particular, in the category of metric spaces, generalized roundness holds on intervals: If a metric space has generalized roundness p_1, then it has generalized roundness p_2 for all $p_2 \in [0, p_1]$. They also gave the first examples of finite dimensional Banach spaces that fail to have positive generalized roundness: For $p > 2$ and $n \geq 3$, $\ell_p^{(n)}$ does not have positive generalized roundness. A number of recent applications of the equivalence of negative type and generalized roundness will be discussed in the subsequent sections.

The rest of the paper is structured as follows. Section 2 investigates results on geometric and topological properties that are detected by the roundness of general metric spaces. For example, geodesic spaces of maximal roundness 2 are contractible, and a compact Riemannian manifold with roundness greater than one must be simply connected (Lafont and Prassidis [43]). Section 3 delineates connections between generalized roundness and the Novikov Conjecture via kernels of negative type. Section 4 is more classical functional analysis where competing notions of nonlinear type, and particularly scaled versions of roundness, are discussed at length. In this section we refrain from saying very much about the notion of metric cotype due to Mendel and Naor [49]. This belies the fact that metric cotype is an *extremely* important development in modern functional analysis. (It just does not quite fit with the flow of our discussion since it is not a generalization of roundness *per se*, but rather a dual notion.) In Section 5 we take a look at uniform Banach groups and a technique for "linearizing" certain uniform homeomorphisms. Finally, in Section 6, we address strict negative type and discuss how it can be used as a device to improve lower bounds on the maximal p-negative type of certain finite metric spaces (such as trees). Throughout the paper we highlight a number of open problems and new directions for future research which we consider to be particularly germane.

2. Roundness Properties of Metric Spaces and Groups

It is a simple application to show that the circle (with the arc length) has trivial roundness. That is, it has maximal roundness 1. This implies that maximal roundness for complicated (non-simply connected) geodesic metric spaces must be trivial.

Theorem 2.1 (Lafont and Prassidis [43]).

(1) *Let (X, d) be a compact geodesic space with globally minimizing geodesic. Then the maximal roundness of (X, d) is* 1.
(2) *Let (X, d) be a geodesic metric space so that, for each $p \in X$ there is a simply connected, geodesically convex neighborhood N_p containing p. That is, if γ is a geodesic joining two points in N_p whose length equals to the distance between the two points, then γ is contained in N_p. Then (X, d) has maximal roundness* 1.
(3) *Let M be a compact non-simply connected manifold. Then M has maximal roundness* 1.

Questions. There are two questions left open from Theorem 2.1.

(1) Let M a compact manifold that contains a globally minimizing closed geodesic. Does M have maximal roundness 1?
(2) What can be said about the maximal roundness of non-compact Riemannian manifolds?

Metric spaces with maximal roundness 2 are clearly quite special. Because of Theorem 2.1, we look at simply connected spaces.

Theorem 2.2 (Lafont and Prassidis [43]). *Let X be a simply connected geodesic metric space of roundness* 2*. Then X is contractible.*

We recall the definition of a CAT(0)-space (Bridson and Haeflinger [14]). Let (X, d) be a geodesic metric space. A *geodesic triangle* $\triangle ABC$ is a collection of three geodesic segments joining the points A, B and C. A *comparison triangle* $\triangle A'B'C'$ is a triangle in the Euclidean plane with the same edge lengths as $\triangle ABC$. Let V and U be two points in $\triangle ABC$ and V', U' the corresponding points in $\triangle A'B'C'$. The metric space (X,d) is CAT(0) if $d(V,U) \leq \|V'-U'\|$ for all geodesic triangles $\triangle ABC$ and all $V,U \in \triangle ABC$. In other words, if the distance between all such V and U in X is smaller than the Euclidean distance between U' and V'.

In the theory of geodesic metric spaces, a similar result as in 2.2 is satisfied for CAT(0)-spaces (Bridson and Haeflinger [14]). That suggests that there might be a connection between CAT(0)-spaces and geodesic metric spaces of roundness 2.

Theorem 2.3. *Let X be a geodesic metric space.*

1. *If X is a CAT(0)-space then X has roundness 2.*
2. *If X is a simply connected metric space of roundness 2, then X is CAT(0).*

Remark 2.4.

1. Part (1) was shown in Lafont and Prassidis ([43]) using comparison properties of quadrilaterals in CAT(0)-spaces.
2. Part (2) was shown in Berg and Nikolaev [10]. See also Berg and Nikolaev [9] for additional background and results along these lines.

Any presentation P of a group G defines a Cayley graph for G, $Cay(G,P)$. The vertices of $Cay(G,P)$ are the elements of G. A pair (g,h) of vertices is joined by an edge if $g^{-1}h$ is a generator or the inverse of a generator. Notice that the combinatorial properties of the Cayley graph of the group depend on the presentation and thus they are not invariants of the group. But the properties of the Cayley graph "at infinity", in "large distances" are algebraic invariants. (See, for example, Bridson and Haeflinger [14], de la Harpe [18], and Gromov [29].) We consider the vertices of $Cay(G,P)$ as a discrete metric space with the ordinary graph metric. In general, the maximal roundness of the Cayley graph of the group depends on the presentation of the group because roundness is not a *quasi-isometric* invariant.

Recall that a (not necessarily continuous) map $f:(X,d_X)\to(Y,d_Y)$ is called a quasi-isometry if there are constants $K>0$, $C>0$ such that, for all $x,y\in X$,

$$\frac{1}{K}d_X(x,y) - C \leq d_Y(f(x),f(y)) \leq K d_X(x,y) + C.$$

Definition 2.5. Let G be a finitely generated group. The *roundness spectrum*, $\rho(G)$, of G is the set of maximal roundness of the Cayley graphs of the group. More precisely,

$$\rho(G) = \{\mathfrak{p}_C :\ C = Cay(G,P),\ P \text{ a presentation of } G\}.$$

We record some basic results on the roundness of Cayley graphs.

Remark 2.6.

1. If G is finite, then $\infty \in \rho(G)$. That is because the complete graph is a Cayley graph of G.
2. If G is a finitely generated infinite group then $\rho(G) \subset [1,2]$.
3. The roundness of an $\mathbb{R}$-tree is 2 (Lafont and Prassidis [43], Naor and Schechtman [53]). Thus $2 \in \rho(F)$, where F is a finitely generated free group.
4. For $n \geq 2$, $\rho(\mathbb{Z}^n) = \{1\}$ (Lafont and Prassidis [43]).

(5) Let G be a finitely generated group that contains two elements x and y that do not have order 2, $x \neq y^{\pm 1}$, $x^3 \neq y^{\pm 1}$, and $y^3 \neq x^{\pm 1}$. Then $1 \in \rho(G)$.
(6) Let G be a finitely generated group and $1 \notin \rho(G)$. Then G is a torsion element with each element of torsion 2, 3, 5 or 7 (Lafont and Prassidis [43]).
(7) It is not known whether or not the roundness spectrum of a group is a closed subset of $[1, \infty)$, in general.

Word Hyperbolic groups generalize the metric properties of the free group. That suggests the following question.

Question. Let G be a word hyperbolic group. Is the roundness spectrum $\rho(G)$ a dense subset of the interval $[1, 2]$?

3. Generalized Roundness and the Novikov Conjecture

The connection between generalized roundness and negative type is given in Theorem 1.10.

We recall the following concept that was introduced by Gromov [29].

Definition 3.1. Let X and Y be metric spaces. A map $f : X \to Y$ is a *coarse embedding* if there are non-decreasing functions $\rho_{\pm} : [0, \infty) \to [0, \infty)$ such that:

$$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y)) \text{ for all } x, y \in X,$$

and with $\lim_{t \to \infty} \rho_-(t) = 0$. Notice that f is not necessarily continuous.

Gromov [29] asked whether every separable metric space admits a coarse embedding into Hilbert space. This question is (in other words) a coarse analog of Smirnov's Question on uniform embeddings. Dranishnikov *et al* [22] constructed a counterexample to *Gromov's Question* by modifying the beautiful ideas of Enflo [24] on uniform embeddings and generalized roundness.

Based on these ideas and results it is natural to make the following definition in analogy to Definition 1.8.

Definition 3.2. A metric space (X, d) is called a *universal coarse embedding space* if for each separable metric space (Y, ρ) there exists a coarse embedding of (Y, ρ) into (X, d).

It (moreover) seems reasonable to pose the following question (to which one might expect a negative answer).

Question. Do there exist universal coarse embedding spaces that have positive generalized roundness?

More generally, is a natural program of study to seek parallels between the theories of uniform and coarse embeddings and, indeed, this has already been done quite extensively. See for example, Nowak [54].

Definition 3.3. Let (X, d) be a metric space on which a group Γ acts by isometries. The Γ-equivariant Hilbert space compression of X, $R_\Gamma(X)$, is the supremum of all β, $0 < \beta < 1$, for which there is a Γ-equivariant coarse embedding f into a Hilbert space on which Γ acts by affine isometries so that $\rho_+(f)$ is affine and $\rho_-(f) = r^\beta$, for large enough r. More precisely, f is a coarse embedding that satisfies $f(\gamma x) = \gamma f(x)$, for all $\gamma \in \Gamma$, $x \in X$.

Remark 3.4.

(1) Jaudon, in [35], proved a quantitative version of Theorem 1.10. More precisely, it is noted that
$$R_\Gamma(X) \geq \frac{\mathfrak{q}_X}{2}.$$
(2) The maximum generalized roundness of the graph corresponding to the standard presentation of the finitely generated free group and free abelian group is computed to be equal to 1 by Jaudon in [35].

The geometric interest in the generalized roundness of a metric space comes from its connections to the coarse Baum–Connes Conjecture. (See Yu [70] for background.) Recall that a discrete metric space (X, d) has bounded geometry if for each $r > 0$, there is a uniform bound $N(r)$ for the cardinality of the balls of radius r.

Coarse Baum–Connes Conjecture. For a discrete metric space X of bounded geometry, the coarse assembly map $\mu : KX_*(X) \to K_*(C^*(X))$ is an isomorphism. Here $KX_*(X)$ denotes the coarse K-homology of X.

If a group, with the word metric, satisfies the coarse Baumm-Connes Conjecture, then it satisfies the Novikov Conjecture, which has a lot of applications to geometric rigidity of topological manifolds.

The following is the basic geometric property of groups with a Cayley graph of positive generalized roundness

Theorem 3.5 (Lafont and Prassidis [43]). *Let Γ be a finitely generated group and X a metric space of positive generalized roundness. Assume that one of the following holds:*

(1) *There is a Cayley graph of Γ that isometrically embeds into X, or*
(2) *Γ acts properly discontinuously, cocompactly and with finite stabilizers by isometries on X.*

Then Γ satisfies the coarse Baum–Connes Conjecture and thus the strong Novikov Conjecture.

Proof. Theorem 1.10 implies that d^p is a negative definite kernel on X. Thus (see Guentner, Higson and Weinberger [30], and Guentner and Kaminker [31]) X coarsely embeds into a Hilbert space. Thus the conditions of Theorem 3.5 imply that Γ coarsely embeds into a Hilbert space. Then, using Yu [70], we deduce that Γ satisfies the coarse Baum–Connes Conjecture and thus the Novikov Conjecture. $\square$

Remark 3.6.

(1) Assume that Γ has a Cayley graph that admits an isometric embedding into $L_p(\mu)$ with $1 \leq p \leq 2$, Then Γ satisfies the coarse Baum–Connes Conjecture and thus the Novikov Conjecture. The result follows because the generalized roundness of $L_p(\mu)$ is equal to p, for $1 \leq p \leq 2$ (Lennard *et al* [45], see also Nowak [54]).
(2) Let Γ be a finitely generated infinite Kazhdan group. Then every negative definite kernel on Γ is bounded (de la Harpe and Valette [18], Delorme [19]). Then, Theorem 1.10 implies that every Cayley graph of Γ has generalized roundness 0.

(3) Let Γ be a uniform lattice in $Sp(n,1)$ or $F_{4(-20)}$. Then Γ is a Kazhdan group ([18]) and thus every Cayley graph of Γ has generalized roundness 0. It is not hard to see that Γ does not satisfy the conditions of Theorem 3.5. But Γ acts isometrically on a quartenionic hyperbolic space or on the Cayley hyperbolic plane, and thus Γ is δ-hyperbolic. But δ-hyperbolic groups embed into a Hilbert space and thus Γ satisfies the coarse Baum–Connes Conjecture and thus the Novikov Conjecture.

Remark 3.6 leaves open the following questions.

Question. Is every CAT(0)-space quasi-isometric to a metric space of positive generalized roundness?

Remark 3.7.

(1) A positive answer to this question will imply that every CAT(0)-group satisfies the Novikov Conjecture.
(2) In Jaudon [35], it is shown that the generalized roundness of the 0-skeleton of a CAT(0)-cubical complex, with the combinatorial metric induced by the 1-skeleton, is at least one.

Question. Let X and Y be discrete metric spaces such that:

(1) The distances are bounded away from 0.
(2) X and Y have bounded geometry.
(3) X and Y are bi-Lipschitz equivalent.

Then is it true that the generalized roundness of X is positive iff the generalized roundness of Y is?

Remark 3.8.

(1) The classical examples of metric spaces X and Y that satisfy the assumptions in the question are Cayley graphs of the same group.
(2) A positive answer to the question will imply that positive generalized roundness is an invariant of the group.
(3) The conditions stated are needed because the maximal generalized roundness of $L_p(\mu)$-spaces (of dimension at least three) is zero for $p > 2$. This result was obtained in Lenard *et al* [45].

Question. Characterize all the groups that admit Cayley graphs of positive generalized roundness.

The groups that appear in the above question will satisfy the coarse Baum–Connes Conjecture. Such a characterization will give us a rich class of groups that satisfy the Novikov Conjecture. It will be interesting to compare this class with the class of groups studied by Kasparov and Yu [40]. The authors believe that there will be more groups in this class. The interesting project will be the study of the metric properties of such groups. More precisely, to what kind of Banach spaces can they be coarsely embedded.

4. Rademacher Type, Metric Type and Scaled Versions of Roundness

Since being isolated in the 1970s the notions of Rademacher type and cotype have become central to any proper understanding of the local theory of Banach spaces. Recall that a Banach space $(X, \|\cdot\|)$ has Rademacher type $p > 0$ if there exists a constant $\mathcal{K} > 0$ such that for any $n \in \mathbb{N}$ and any $x_1, \ldots, x_n \in X$ we have

$$\frac{1}{2^n} \sum_{\varepsilon} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|^p \quad \leq \quad \mathcal{K}^p \cdot \sum_{i=1}^{n} \|x_i\|^p \tag{4}$$

where the summing on the left is taken over all $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, +1\}^n$. Cotype is defined analogously but with the above inequalities reversed. These are linear Banach space notions insofar as the definitions involve both addition and scalar multiplication of vectors. Already by the mid 1970s there were serious calls for metric (or, nonlinear) versions of fundamental local properties of Banach spaces such as type and cotype. In part, these calls were driven by Ribe's [58] discovery that uniformly homeomorphic Banach spaces must have the same local structure. More precisely, Ribe proved that if X and Y are uniformly homeomorphic Banach spaces, then X is finitely representable in Y, and *vice versa*. An unpublished result of Enflo[3] shows that converse of Ribe's theorem does not hold in general. Enflo considered $L_1[0,1]$ and ℓ_1. These Banach spaces have the same local structure but they are not uniformly homeomorphic.

Initial candidates for metric versions of type were introduced by Enflo [27] and Bourgain *et al* [12], respectively. Both notions are predicated in terms of nonlinear n-cubes (see Definition 1.4) and are defined as follows.

Definition 4.1. Let (X, d) be an (infinite) metric space. Let $p \geq 1$.

(a) (X, d) has *Enflo type* p if there is a constant $\mathcal{E} > 0$ such that for every $n \in \mathbb{N}$ and every n-cube $N = \{x_\varepsilon\}$ in X we have:

$$\left(\sum_{d \in D(N)} l(d)^p \right)^{\frac{1}{p}} \quad \leq \quad \mathcal{E} \cdot \left(\sum_{e \in E(N)} l(e)^p \right)^{\frac{1}{p}}. \tag{5}$$

(b) (X, d) has *BMW type* (or, *metric type*) p if there is a constant $\mathcal{B} > 0$ such that for every $n \in \mathbb{N}$ and every n-cube $N = \{x_\varepsilon\}$ in X we have:

$$\left(\sum_{d \in D(N)} l(d)^2 \right)^{\frac{1}{2}} \quad \leq \quad \mathcal{B} \cdot n^{\frac{1}{p} - \frac{1}{2}} \cdot \left(\sum_{e \in E(N)} l(e)^2 \right)^{\frac{1}{2}}. \tag{6}$$

A Banach space $(X, \|\cdot\|)$ that has Enflo type p must have Rademacher type p. Indeed, given $x_1, \ldots, x_n \in X$, one simply considers the n-cube $N = \{x_\varepsilon\} \subset X$, where $x_\varepsilon = \varepsilon_1 x_1 + \cdots + \varepsilon_n x_n$ for each $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, +1\}^n$, to obtain that (5) implies (4). Similarly, BMW type p implies Rademacher type p for Banach spaces. The following natural question was posed by Enflo [27] and remains open in full generality.

Question. If a Banach space has Rademacher type p must it have Enflo type p?

Pisier [56] showed that if a Banach space has Rademacher type p, then it has Enflo type q for all $q < p$. The corresponding result where Enflo type q is replaced by BMW type q was obtained by Bourgain *et al* [12] prior to Pisier's work. Hence the supremum of the Rademacher types of a Banach space cannot exceed — and therefore must equal — the supremum of its Enflo (or, BMW) types. Naor and

[3] See Case (i) in the proof of Theorem 10.13 in Benyamini and Lindenstrauss [8] for Enflo's original argument.

Schechtman [53] have recently shown that Rademacher type p is equivalent to Enflo type p for UMD Banach spaces. (We recall briefly that UMD spaces are Banach spaces in which martingale differences are unconditional. See, for example, Naor and Schechtman [53] or Burkholder [15] for more precise information on UMD spaces.) Pisier [56] also showed that if a metric space has Enflo type p, then it has BMW type q for all $q < p$. Whether or not the converse of this statement holds for metric spaces appears to be an open problem.

The next result gives a natural extension of Theorem 1.6 to the realm of BMW type. We include the proof since it gives a nice illustration of working with n-cubes and nonlinear roundness (in this case, BMW type).

Theorem 4.2 (Lennard *et al* [46])**.** *Suppose $(X, \|\cdot\|)$ is an infinite-dimensional Banach space that contains the finite dimensional spaces ℓ_p^n uniformly in $n \in \mathbb{N}$. Let (Y, ρ) be a metric space with BMW type $q > p$, and let $T : (X, \|\cdot\|) \to (Y, \rho)$ be an onto uniform homeomorphism. Then T^{-1} cannot satisfy a Lipschitz condition of order α for large distances for any $\alpha \in (0, \frac{q}{p})$.*

Proof. Since the map T is uniformly continuous on X it must be Lipschitz of order one for large distances by the Corson-Klee Lemma. It follows that the quantity

$$K = \sup\{\, \rho(Tx, Ty) \,|\, x, y \in X \text{ and } \|x - y\| \leq 1 \,\}$$

is finite. By hypothesis, there exists $\mathcal{M} \in (0, 1)$ and, for each $n \in \mathbb{N}$, a sequence $(e_j^{(n)})_{j=1}^n$ in X such that

$$\mathcal{M}\left(\sum_{j=1}^{n} |\alpha_j|^p\right)^{\frac{1}{p}} \leq \left\|\sum_{j=1}^{n} \alpha_j e_j^{(n)}\right\| \leq \left(\sum_{j=1}^{n} |\alpha_j|^p\right)^{\frac{1}{p}}$$

for all scalar sequences $(\alpha_j)_{j=1}^n$.

Fix a natural number n. Let $x_\xi^{(n)} = \sum_{j=1}^n \xi_j e_j^{(n)}$, for all $\xi \in \{0,1\}^n$. The family $C_n = (x_\xi^{(n)})_{\xi \in \{0,1\}^n}$ is an n-cube in X, with $l(e) \in [\mathcal{M}, 1]$ for all $e \in E(C_n)$, and $l(d) \in [\mathcal{M}n^{\frac{1}{p}}, n^{\frac{1}{p}}]$ for all $d \in D(C_n)$. The image of C_n under T, namely $R_n = (Tx_\xi^{(n)})_{\xi \in \{0,1\}^n}$, is an n-cube in (Y, ρ).

Since (Y, ρ) has metric type q it follows that

$$\left(\sum_{d \in D(R_n)} l(d)^2\right)^{\frac{1}{2}} \leq \mathcal{B} n^{\frac{1}{q} - \frac{1}{2}} \left(\sum_{e \in E(R_n)} l(e)^2\right)^{\frac{1}{2}}$$

for some $\mathcal{B} \in (0, \infty)$.

Now suppose that T^{-1} satisfies a Lipschitz condition of order α for large distances. Then, for all $\delta > 0$, there exists a constant $\Gamma = \Gamma(\delta) \in (0, \infty)$ such that $\|T^{-1}u - T^{-1}v\|_X \leq \Gamma(\delta)\rho(u, v)^\alpha$ whenever $\rho(u, v) \geq \delta$ $(u, v \in Y)$. But because T^{-1} is uniformly continuous and $l(d) \geq \mathcal{M}n^{\frac{1}{p}}$ for all $d \in D(C_n)$, it follows that

$$\gamma = \inf_{n \in \mathbb{N}} \min_{d \in D(R_n)} l(d)$$

is a positive number. Let $\Gamma = \Gamma(\gamma)$ and again fix $n \in \mathbb{N}$. Then

$$\left(\sum_{d \in D(C_n)} (\Gamma^{-1} l(d))^{\frac{2}{\alpha}}\right)^{\frac{1}{2}} \leq \mathcal{B} n^{\frac{1}{q} - \frac{1}{2}} (n 2^{n-1} K^2)^{\frac{1}{2}}$$

and so

$$2^{\frac{n-1}{2}}(\Gamma^{-1}\mathcal{M})^{\frac{1}{\alpha}}\, n^{\frac{1}{p\alpha}} \leq \mathcal{B}n^{\frac{1}{q}}2^{\frac{n-1}{2}}K.$$

Since $n \in \mathbb{N}$ is arbitrary, we conclude that $\frac{q}{p} \leq \alpha$. $\square$

Despite the close connections between Rademacher type, Enflo type and BMW type in the category of Banach spaces, a purely metric formulation of Rademacher type was only obtained very recently. Mendel and Naor [50], motivated by their work on metric cotype [49], introduced a scaled version of Enflo type p which is equivalent to Rademacher type p for Banach spaces. The precise definition of scaled Enflo type is as follows.

Definition 4.3. Let $\mathbb{E}_\varepsilon$ denote the expectation with respect to uniformly chosen vectors $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in \{-1, +1\}^n$. Let (X, d_X) be a metric space and let $p > 0$. We say that (X, d_X) has *scaled Enflo type* p with constant $\mathcal{M}$ if for every integer n there exists an even integer m such that for every function $f : \mathbb{Z}_m^n \to X$, we have

$$\mathbb{E}_\varepsilon \int_{\mathbb{Z}_m^n} d_X\left(f\big(x + \frac{m}{2}\varepsilon\big), f(x)\right)^p d\mu(x) \quad \leq \quad \mathcal{M}^p m^p \sum_{j=1}^{n} \int_{\mathbb{Z}_m^n} d_X(f(x+e_j), f(x))^p d\mu(x),$$

where μ is the uniform probability measure on $\mathbb{Z}_m^n$ and $\{e_j\}_{j=1}^n$ is the standard basis of $\mathbb{R}^n$.

Mendel and Naor [50] point out a number of relationships between Enflo type and scaled Enflo type including the following. If a metric space has Enflo type p, then it must have scaled Enflo type p. It is not known at this time if the converse of this statement holds for general metric spaces. However, for Banach spaces, scaled Enflo type p implies Enflo type q for all $q < p$. Moreover, scaled Enflo type p and Enflo type p coincide for UMD Banach spaces.

Another version of nonlinear type that has proven to be very important in understanding the nonlinear geometry of metric spaces is Ball's [6] notion of Markov type. Recall that a Markov chain $(Z_t)_{t=0}^\infty$ with transition probabilities $a_{ij} = \Pr(Z_{t+1} = j | Z_t = i)$ on the state space $[n] = \{1, 2, \ldots, n\}$ is *stationary* if $\pi_i = \Pr(Z_t = i)$ does not depend on t and *reversible* if $\pi_i a_{ij} = \pi_j a_{ji}$ for every $i, j \in \{1, 2, \ldots, n\}$.

Definition 4.4. Let $p \geq 1$ and let (X, d) be a metric space. We say that (X, d) has *Markov type* p if there exists a constant $\mathcal{M} > 0$ such that for every stationary reversible Markov chain $(Z_t)_{t=0}^\infty$ on the state space $[n]$, every mapping $f : [n] \to X$ and every time $t \in \mathbb{N}$, we have:

$$\mathbb{E}\big(d(f(Z_t), f(Z_0))^p\big) \quad \leq \quad \mathcal{M}^p \cdot t \cdot \mathbb{E}\big(d(f(Z_1), f(Z_0))^p\big).$$

Ball [6] introduced Markov type in a deep study of the extension problem for Lipschitz maps and obtained the following celebrated result: any Lipschitz map from a subset of a metric space (X, d) having Markov type 2 into a Banach space $(Y, \|\cdot\|)$ with modulus of convexity of power type 2 can be extended to a Lipschitz map on the entire space (X, d).

At first, only Hilbert space and metric spaces that bi-Lipschitzly embed into Hilbert space were known to have Markov type 2. However, Naor *et al* [52] have since shown that Banach spaces with modulus of smoothness of power type 2 (examples include the $L_p(\mu)$-spaces for $p > 2$), metric trees, hyperbolic groups and simply connected Riemannian manifolds of pinched negative curvature all have Markov type 2. Further, Ohta [55] has established that Alexandrov spaces of non-negative curvature have Markov type 2. It also follows from negative type considerations and Theorem 1.9 that $L_p(\mu)$-spaces have Markov type p for $p \in [1,2]$.

Naor and Schechtman [53] have shown that for general metric spaces, Markov type p implies Enflo type p. However, very little seems to be known about the interplay between Markov type and Rademcher type for Banach spaces. The following question therefore seems natural.

Question. For which metric spaces does Enflo type p (or, alternatively, BMW type p) imply Markov type p?

The notions of roundness and Markov type have also been shown to have ramifications in the analysis of the Hilbert space compression (exponent) of finitely generated discrete groups. This notion was introduced by Guentner and Kaminker [32] as a quasi-isometric invariant for groups and may be defined in the following way.

Definition 4.5. Let G be a finitely generated discrete group. Let $S \subseteq G$ be a fixed finite and symmetric ($S = S^{-1}$) set of generators for G. Let d denote the left-invariant word metric induced by S on G. The *Hilbert space compression (exponent)* of G is defined to be the supremum over all $\alpha \geq 0$ for which there exists a 1-Lipschitz mapping $f : G \to L_2$ together with constants $c_1, c_2 > 0$ such that:

$$\|f(x) - f(y)\|_2 \geq c_1 d(x,y)^\alpha - c_2$$

for all $x, y \in G$.

Guentner and Kaminker [32] established that groups with Hilbert space compression strictly greater than $\frac{1}{2}$ necessarily have Yu's [70] *Property A*. They also determined that free groups have Hilbert space compression 1.

The first example of a group with Hilbert space compression strictly between 0 and 1 was given by Arzhantseva *et al* [4] who showed that the Hilbert space compression of F. Thompson's group F equals $\frac{1}{2}$.[4] In proving this result the authors used Theorem 1.5 in the case $p = 2$ (in other words, that Hilbert space has roundness 2) in a critical way. Arzhantseva *et al* [4] also obtained that the Hilbert space compression of the restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$ lies in the interval $[\frac{1}{2}, \frac{3}{4}]$. The lower bound of $\frac{1}{2}$ on the Hilbert space compression of $\mathbb{Z} \wr \mathbb{Z}$ was later improved to $\frac{2}{3}$ by Tessera [65]. Finally, Austin *et al* [5] have shown that the Hilbert space compression of $\mathbb{Z} \wr \mathbb{Z}$ is exactly $\frac{2}{3}$ as an application of Markov type.

5. Uniform Banach Groups

The notion of a uniform Banach group generalizes the additive group structure of a Banach space. It is a group structure on a Banach space that satisfies some compatibility conditions with the norm. Uniform Banach groups were introduced and studied extensively by Enflo in [25] and [26].

Definition 5.1. Let X be a Banach space. Suppose that the map $X \times X \to X : (x, y) \mapsto x \cdot y$ 'is a group operation on X that is uniformly continuous as a function of two variables with identity $0 = 0_X$, the zero vector of X. Then the resulting

[4]Recall that R. Thompson's group F is the group of all piecewise linear orientation preserving self-homeomorphisms of the unit interval with finitely many dyadic singularities and all slopes integer powers of 2.

group structure $G = (X, \cdot)$ is said to be a *uniform Banach group modelled on X*. (We often write xy instead of $x \cdot y$.)

Uniform homeomorphisms generate examples of uniform Banach groups in the following way: suppose that X is a Banach space, Y is a topological vector space, and $\phi : X \to Y$ is a uniform homeomorphism with $\phi(0) = 0$. Then the product $x \cdot y = \phi^{-1}(\phi(x) + \phi(y))$ defines a commutative uniform Banach group structure $G = (X, \cdot)$ that is modelled on X. We refer to G as being the *uniform Banach group induced by ϕ*.

Question. Do there exist *noncommutative* uniform Banach groups?

Remark 5.2. Natural examples of noncommutative groups which are modelled on some Banach space (in this case $\mathbb{C}^n \times \mathbb{R}$ regarded as a real Banach space) are the classical Heisenberg groups $\mathbb{H}^{2n+1}$, $n \in \mathbb{N}$. Indeed, given $n \in \mathbb{N}$, the $2n+1$ dimensional Heisenberg group $\mathbb{H}^{2n+1}$ consists of the noncommutative group operation

$$(z, s) \cdot (w, t) = \left(z + w, t + s + 2 \sum_{j=1}^{n} \Im(z_j \overline{w_j}) \right)$$

defined on $\mathbb{C}^n \times \mathbb{R}$ (where, as one would expect, $z = (z_j)_{j=1}^n$ and $w = (w_j)_{j=1}^n$ are in $\mathbb{C}^n$, and so on). Clearly the identity element of the Heisenberg group $\mathbb{H}^{2n+1}$ is the zero vector. However, despite having many nice properties, the group operation on H^{2n+1} is not *uniformly* continuous as a function of two variables. This is simply because it involves the multiplication of arbitrary complex numbers, and hence cannot be uniformly continuous (unless restricted to a bounded subset of $\mathbb{C}^n \times \mathbb{R}$). Hence the Heisenberg groups $\mathbb{H}^{2n+1}$ are not examples of noncommutative uniform Banach groups.

Through an analysis of the interplay between non-trivial roundness and commutative uniform Banach groups, Enflo [26] obtained Theorem 1.7 which we discussed briefly in the introduction to this paper. In particular, Enflo [26] noted that $L_{p_1}[0,1]$ is not uniformly homeomorphic to $L_{p_2}[0,1]$ if $0 < p_1 < 1 < p_2 < \infty$. Lurking beneath the surface of Enflo's arguments was a linearization procedure for certain uniform homeomorphisms. This procedure was isolated (and generalized to include a consideration of non-trivial metric type) in papers by Prassidis and Weston [57], and Weston [69]. As linearization procedures are relatively uncommon in the uniform theory of Banach spaces, we include a fairly complete description of this technique and indicate an outstanding open question which does not yield to this approach.

Suppose henceforth that $G = (X, \cdot)$ is a commutative uniform Banach group modelled on $(X, \|\cdot\|)$. We can introduce a G-invariant metric d on X as follows:

$$d(x, y) = \sup_{w \in X} \|wx - wy\| \, ; \, x, y \in X.$$

Obviously $d(x, y) \geq \|x - y\|$ for all $x, y \in X$. As is easy to show, this G-invariant metric d is uniformly equivalent to the $\|\cdot\|$-distance. In other words, the identity map $(X, \|\cdot\|) \to (X, d) : x \mapsto x$ is a uniform homeomorphism. As d is a G-invariant metric on X, we get a triangle inequality of the form $d(xy, 0) \leq d(x, 0) + d(y, 0)$ for all $x, y \in X$. Associated with the G-invariant metric d is a *chain* or *intrinsic metric* d_I defined in the following way. Let $x, y \in X$ be given. Any finite sequence $x = x_0, x_1, \ldots, x_n = y$ ($n \in \mathbb{N}$) of points in X such that $d(x_i, x_{i+1}) \leq 1$ for all

i, $0 \leq i \leq n-1$, is called a *one-chain* between x and y. The *intrinsic* or *chain distance* between x and y is given by:

$$d_I(x,y) = \inf_{\mathcal{C}} \sum_{i=0}^{n-1} d(x_i, x_{i+1}),$$

where the infimum is taken over all one-chains $x = x_0, x_1, \ldots, x_n = y$ between x and y. For all $x, y \in X$ it is clear that $\|x-y\| \leq d(x,y) \leq d_I(x,y)$ and — moreover — that $d_I(x,y) = d(x,y)$ if (additionally) $d(x,y) \leq 1$. It follows that the chain metric d_I is also uniformly equivalent to the $\|\cdot\|$-distance.

Every Banach space has (metric) type 1. In the statement of Theorem 5.3 (below) the essential hypothesis is on the Banach space X. Namely, that it has non-trivial (metric) type $p > 1$. The assumption that Y is a Banach space can be weakened considerably. This is done in Theorem 5.4. The technique being described in this section is seen to have two important aspects; (i) Y is not restricted to be a normed (or, even quasi-normed) space, and consequently (ii) the Corson-Klee Lemma is not being used (since it does not apply to such spaces in general).

Theorem 5.3. *Let $(X, \|\cdot\|)$ be a Banach space with metric type $p > 1$. If $(X, \|\cdot\|)$ is uniformly homeomorphic to a Banach space $(Y, |||\cdot|||)$, then the vector space operations and norm on X can be re-modelled (in a uniformly equivalent way) to produce a new Banach space $(X, N(\cdot))$ that has metric type p, and which is (moreover) linearly isomorphic to $(Y, |||\cdot|||)$. In particular, Y is seen to have metric type p.*

Proof: Linearization of Certain Uniform Homeomorphisms Technique. We may assume that the uniform homeomorphism $\phi : (X, \|\cdot\|) \to (Y, |||\cdot|||)$ satisfies $\phi(0) = 0$. Denote by $G = (X, \cdot)$ the uniform Banach group modelled on $(X, \|\cdot\|)$ that is induced by ϕ. So, for all $x, y \in X$, $xy = \phi^{-1}(\phi(x) + \phi(y))$. Introduce the G-invariant metrics d and d_I associated with this group structure. For any $x \in X$ and $t \in \mathbb{R}$, define $x^t = \phi^{-1}(t\phi(x)$. Introduce new vector space operations on X as follows: addition is the group multiplication $(x,y) \mapsto xy$, and scalar multiplication of $x \in X$ by $t \in \mathbb{R}$ is given by x^t. For all $x \in X$, define:

$$N(x) = \limsup_{t \to \infty} \frac{d_I(x^t, 0)}{t}.$$

Prassidis and Weston [57] (Theorem 4.2) have shown that $N(\cdot)$ is a norm relative to the new vector space operations on X, and that

$$\|x\| \leq N(x) \leq d_I(x, 0) \tag{7}$$

for all $x \in X$.[5] For brevity we will write $(X, N(\cdot))$ to denote this new normed vector space structure. A simple (omitted) argument shows that $(X, N(\cdot))$ is complete. Since d_I is uniformly equivalent to the original $\|\cdot\|$-distance, it follows from (7) that the identity map $(X, \|\cdot\|) \to (X, N(\cdot)) : x \mapsto x$ is a (not necessarily linear) uniform homeomorphism. Hence, by virtue of the new vector space operations in place on X, we see that $\phi : (X, N(\cdot)) \to (Y, |||\cdot|||)$ is a linear uniform homeomorphism. Put differently, as a map from $(X, N(\cdot))$ to $(Y, |||\cdot|||)$, ϕ is a linear isomorphism.

The argument that $(X, N(\cdot))$, and hence $(Y, |||\cdot|||)$, has metric type p is relatively simple. The details may be found in Weston [69]. $\square$

[5] We remark that the derivation of (7) depends explicitly on the Banach space $(X, \|\cdot\|)$ having metric type $p > 1$, a condition which we have not been able to relax in this context.

As noted above, the hypothesis on Y in the statement of Theorem 5.3 can be greatly relaxed. In fact, the next theorem determines that Y need only be a topological vector space.

Theorem 5.4 (Prassidis and Weston [57], Weston [69])**.** *Let Y be a topological vector space. If Y is uniformly homeomorphic to a closed subspace of a Banach space $(X, \|\cdot\|)$ that has non-trivial type, then the topology of Y is given by a norm.*

Proof. The properties of non-trivial type $p > 1$ and completeness are inherited by the closed subspaces of $(X, \|\cdot\|)$. So, without loss of generality, it suffices to consider a uniform homeomorphism $\phi : (X, \|\cdot\|) \to Y$ with $\phi(0) = 0$. The same argument as given in the proof of Theorem 5.3 constructs the remodelled Banach space $(X, N(\cdot))$ that is induced by ϕ. And again, as a map from the Banach space $(X, N(\cdot))$ to the topological vector space Y, ϕ is a linear uniform homeomorphism. (In particular, as X is metrizable, this implies that the map $\phi : (X, N(\cdot)) \to Y$ is bounded. See, for example, Rudin [61] (Theorem 1.32).) It follows from the Open Mapping Theorem (see, for example, Theorem 2.11 in Rudin [61]) that Y is an F-space. (And so the map $\phi^{-1} : Y \to (X, N(\cdot))$ is also bounded by Theorem 1.32 in Rudin [61].) But an F-space that is linearly isomorphic to a Banach space is obviously locally convex and locally bounded. Therefore Y is normable. $\square$

It should be noted that Theorem 5.4 is due to Bessaga [11] in the case that Y is assumed to be a *locally convex* topological vector space. In fact, Bessaga [11] showed that if a locally convex topological vector space is uniformly homeomorphic to a normed space, then it is normable. Theorem 5.4 shows that a non normable topological vector space cannot be uniformly homeomorphic to any Banach space that has non-trivial type. This leaves open the following important question.

Question. Can a non normable topological vector space be uniformly homeomorphic to a Banach space whose supremal Rademacher type is 1?

New examples of uniform non-equivalence can be deduced from Theorem 5.4. This is because the (supremal) Rademacher type of many Banach spaces has been computed. The following families of Banach spaces are all known to have type greater than one: commutative $L_p(\mu)$-spaces with $p \in (1, \infty)$, Schatten-von Neumann classes C_p and other non-commutative L_p-spaces with $p \in (1, \infty)$, Lorentz $L_{p,q}$-spaces if both $p, q \in (1, \infty)$, and Orlicz spaces L_Φ with an appropriate condition on the Orlicz function Φ (see Corollary 5.5 below). Examples of non-normable quasi-normed spaces include the various $L_p(\mu)$ and H_p spaces with $p \in (0, 1)$, and Orlicz spaces L_Ψ with an appropriate condition on the Orlicz function Ψ.

Corollary 5.5 (Prassidis and Weston [57])**.** *Consider locally bounded Orlicz spaces L_Φ and L_Ψ corresponding to (possibly different) non-atomic finite measure spaces. If there is a constant K and a $p > 1$ such that*

$$\Phi(\lambda s) \geq K \cdot \lambda p' \cdot \Phi(s)$$

for $s \geq 0$ whenever $\lambda \geq 1$ and if $\liminf\limits_{t\to\infty} \frac{\Psi(t)}{t} = 0$, then L_Φ is not uniformly homeomorphic L_Ψ.

Proof. L_Φ has type p by a theorem of Kamińska and Turett [39]. L_Ψ is non-normable since it has a trivial dual by a theorem of Rolewicz [60]. Hence these spaces are not uniformly homeomorphic by Theorem 5.4. $\square$

Another such result, that stands out in relation to the uniform classification of L_p-spaces, is the following.

Corollary 5.6 (Weston [69])**.** *Let μ be a positive Borel measure on some sigma algebra (Ω, Σ). If X is a Banach space that has non-trivial type, then the classical F-space $L_0(\mu)$ is not uniformly homeomorphic to any (closed) subspace of X.*

In contrast to Corollary 5.6, Aharoni, Maurey and Mitjagin [1] had previously shown that $L_0(\mu)$ is however uniformly homeomorphic to a *subset* of Hilbert space. More recently, Corollary 5.6 has been generalized by Albiac [3] as follows.

Theorem 5.7 (Albiac [3])**.** *Let X be a locally bounded topological vector space. If $\phi : L_0[0,1] \to X$ is a uniformly continuous map then the range of ϕ is a bounded set in X. In particular, $L_0[0,1]$ is not uniformly homeomorphic to X.*

It should be noted that Albiac's result on $L_0[0,1]$ was obtained without appeal to uniform Banach groups, but rather through a direct and elegant argument specific to $L_0[0,1]$.

Given Banach spaces X and Y of non-trivial (metric) type, and a uniform homeomorphism $\phi : X \to Y$ (that may be assumed to satisfy $\phi(0) = 0$), the linearization procedure described in the proof of Theorem 5.3 is clearly reversible. We can apply the same method to Y and ϕ^{-1} to produce a new Banach space $(Y, M(\cdot))$ that is linearly isomorphic to X and has the same metric types as Y. Hence the supremum of the types of X (which is equal to the supremum of its metric types by Bourgain, Milman and Wolfson [12]) equals the supremum of the types of Y. Moreover, as type is inherited by subspaces, this same argument extends to any (closed) subspace Z of Y that may also happen to be uniformly homeomorphic to X. In summary, we have determined a new proof of the following classical result in the uniform theory of Banach spaces, a result that is essentially due to Ribe [58].

Corollary 5.8 (Ribe [58])**.** *Let $p > 1$. Let X, Y be Banach spaces. Suppose X is uniformly homeomorphic to a (closed) subspace Z of Y. Then X has type p if and only if Z has type p. Put differently, supremal type is a uniform invariant for Banach spaces.*

Several other classical results in the uniform theory of Banach spaces can be deduced from Theorem 5.3, Theorem 5.4 and Corollary 5.8. See, for example, the discussion in Weston [69].

6. Strict negative type and the geometry of finite metric spaces

Recall from Definition 1.1 that a metric space (X, d) has *strict p-negative type* if it has p-negative type and if the non trivial p-negative type inequalities for X are all strict. Finite metric spaces of strict 1-negative type have been studied extensively in papers by Hjorth *et al* [33], [34]. Hjorth *et al* [34] have shown that all finite metric trees have strict 1-negative type.[6] They also show that finite

[6] Recall that a *finite metric tree* is a finite connected graph that has no cycles, endowed with an edge weighted path metric. The study of trees as mathematical objects was initiated by Cayley [16] who enumerated the isomers of the saturated hydrocarbons C_nH_{2n+2}. For example, an application of Cayley's formula shows that the number of isomers of the paraffin $C_{13}H_{28}$ is 802. More recently, mathematical studies of finite metric trees have proliferated due to myriad applications in areas as seemingly diverse as evolutionary biology and theoretical computer science. Some examples of publications which highlight this point include Weber *et al.* [66], Ailon and Charikar [2], Semple and Steel [64], Fakcharoenphol *et al.* [28], Charikar *et al.* [17], and Bartal [7].

subsets of spheres (endowed with the natural inherited geodesic metric) which do not contain more than one pair of antipodal points have strict 1-negative type. This is actually for the same reason that the circle with arc length has maximal roundness 1, so one can compare such a result with Theorem 2.1. In a sequel, Hjorth *et al* [33] elaborate conditions on Riemannian manifolds endowed with the Riemannian distance function which ensure strict 1-negative type.

Theorem 6.1 (Hjorth *et al* [33]). *Let M be a Riemannian manifold equipped with the Riemannian distance function d. If $d : M \times M \setminus \{(p,p) | p \in M\} \to \mathbb{R}^+$ is C^1 and if (M,d) has 1-negative type, then (M,d) necessarily has strict 1-negative type. It follows (for example) that the hyperbolic spaces $\mathbb{H}^n_{\mathbb{R}}$ and $\mathbb{H}^n_{\mathbb{C}}$ have strict 1-negative type. More generally, any Hadamard manifold has strict 1-negative type.*

The next result bears a relationship to Theorem 2.1 (3).

Theorem 6.2 (Hjorth *et al* [33]). *A compact Riemannian manifold (M,d) of negative type is simply connected.*

The converse of Theorem 6.2, however, is not true in general.

Theorem 6.3 (Hjorth *et al* [33]). *A Riemannian product manifold with a sphere $\mathbb{S}^n$ (any $n \geq 1$) as a factor cannot have 1-negative type.*

In so far as strict 1-negative type has been relatively well studied, properties of strict p-negative type for $p \neq 1$ remain obscure and there are a number of intriguing open problems which we would like to discuss. Perhaps the only paper with an extensive study of strict p-negative type for $p \neq 1$ is Doust and Weston [21]. Before addressing some of the results of this paper we will make some preliminary comments. We begin with an expected connection to generalized roundness to whit the following definition is helpful.

Definition 6.4. Let X be a set. Let q, t be natural numbers.

(a) A (q,t)-*simplex* in X is a $(q+t)$-vector $(a_1, \ldots, a_q, b_1, \ldots, b_t) \in X^{q+t}$ whose coordinates consist of $q+t$ distinct vertices $a_1, \ldots, a_q, b_1, \ldots, b_t \in X$. Such a simplex will be denoted $D = [a_j; b_i]_{q,t}$.

(b) A *load vector* for a (q,t)-simplex $D = [a_j; b_i]_{q,t}$ in X is an arbitrary vector $\vec{\omega} = (m_1, \ldots m_q, n_1, \ldots, n_t) \in \mathbb{R}^{q+t}_+$ that assigns a positive weight $m_j > 0$ or $n_i > 0$ to each vertex a_j or b_i of D, respectively.

(c) A *loaded* (q,t)-*simplex* in X consists of a (q,t)-simplex $D = [a_j; b_i]_{q,t}$ in X together with a load vector $\vec{\omega} = (m_1, \ldots, m_q, n_1, \ldots, n_t)$ for D. Such a loaded simplex will be denoted $D(\vec{\omega})$ or $[a_j(m_j); b_i(n_i)]_{q,t}$ as the need arises.

(d) A *normalized* (q,t)-*simplex* in X is a loaded (q,t)-simplex $D(\vec{\omega})$ in X whose load vector $\vec{\omega} = (m_1, \ldots, m_q, n_1, \ldots, n_t)$ satisfies the two normalizations:

$$m_1 + \cdots + m_q = 1 = n_1 + \cdots n_t.$$

Such a vector $\vec{\omega}$ will be called a *normalized load vector* for D.

The next theorem characterizes generalized roundness p—or, equivalently, p-negative type—in terms of normalized (q,t)-simplexes.

Theorem 6.5. *Let $p \geq 0$. For a metric space (X,d), the following two conditions are equivalent:*

(a) (X,d) *has generalized roundness* p.
(b) *For all* $q,t \in \mathbb{N}$ *and all normalized* (q,t)*-simplexes* $D(\vec{\omega}) = [a_j(m_j); b_i(n_i)]_{q,t}$ *in* X *we have:*

$$\sum_{1 \leq j_1 < j_2 \leq q} m_{j_1} m_{j_2} d(a_{j_1}, a_{j_2})^p + \sum_{1 \leq i_1 < i_2 \leq t} n_{i_1} n_{i_2} d(b_{i_1}, b_{i_2})^p \leq \sum_{j,i=1}^{q,t} m_j n_i d(a_j, b_i)^p. \tag{8}$$

Remark 6.6. The weighted generalized roundness inequalities of Theorem 6.5 (b) were first introduced by Weston [68] who used them to show that every finite metric space (X,d) has positive generalized roundness p where, moreover, $p > 0$ may be chosen so as to only depend upon $|X|$. It is clear in Theorem 6.5 that condition (b) implies condition (a). To establish the converse one allows repititions among the a's and the b's in order to introduce positive integer weights. (Compare with Remark 1.2.) One may then normalize the resulting inequality to obtain (8) with positive rational weights, and then use simple continuity arguments to pass to normalized (q,t)-simplexes. A similar style of argument was used by Lennard *et al* [45] to prove that generalized roundness p implies p-negative type. (That is, condition (b) implies condition (a) in the statement of Theorem 1.10.)

The advantage of working with Theorem 6.5 (b)—instead of generalized roundness p or p-negative type—is that trivial cases of equality in the inequalities (8) are automatically excluded. Motivated by this observation this we make the following natural definition.

Definition 6.7. Let $p \geq 0$. A metric space (X,d) has *strict generalized roundness* p if and only if for all $q,t \in \mathbb{N}$ and all normalized (q,t)-simplexes $D(\vec{\omega}) = [a_j(m_j); b_i(n_i)]_{q,t}$ in X we have:

$$\sum_{1 \leq j_1 < j_2 \leq q} m_{j_1} m_{j_2} d(a_{j_1}, a_{j_2})^p + \sum_{1 \leq i_1 < i_2 \leq t} n_{i_1} n_{i_2} d(b_{i_1}, b_{i_2})^p < \sum_{j,i=1}^{q,t} m_j n_i d(a_j, b_i)^p. \tag{9}$$

The following equivalence was noted by Doust and Weston [21] who used it as a device to help understand strict p-negative type for the less well travelled cases $p \neq 1$.

Theorem 6.8. *Let* $p \geq 0$. *A metric space* (X,d) *has strict* p*-negative type if and only if it has strict generalized roundness* p.

Based on the above definitions and theorems we isolate two parameters $\gamma_D^p(\vec{\omega})$ and Γ_X^p that are designed to *vicariously* quantify the "degree of strictness" of the non trivial p-negative type inequalities. The two relevant definitions are as follows.

Definition 6.9. Let $p \geq 0$. Let (X,d) be a metric space. Let q,t be natural numbers. Let $D = [a_j; b_i]_{q,t}$ be a (q,t)-simplex in X. Let $N_{q,t} \subset \mathbb{R}_+^{q+t}$ denote the set of all normalized load vectors $\vec{\omega} = (m_1, \ldots, m_q, n_1, \ldots, n_t)$ for D. Then, the *p-negative type simplex gap* of D is the function $\gamma_D^p : N_{q,t} \to \mathbb{R} : \vec{\omega} \mapsto \gamma_D^p(\vec{\omega})$ where:

$$\begin{aligned}\gamma_D^p(\vec{\omega}) &= \sum_{j,i=1}^{q,t} m_j n_i d(a_j, b_i)^p \\ &\quad - \sum_{1 \leq j_1 < j_2 \leq q} m_{j_1} m_{j_2} d(a_{j_1}, a_{j_2})^p - \sum_{1 \leq i_1 < i_2 \leq t} n_{i_1} n_{i_2} d(b_{i_1}, b_{i_2})^p,\end{aligned}$$

for each $\vec{\omega} = (m_1, \ldots, m_q, n_1, \ldots, n_t) \in N_{q,t}$. If we further define the quantities

$$\begin{aligned}\mathfrak{R}_D^p(\vec{\omega}) &= \sum_{j,i=1}^{q,t} m_j n_i d(a_j, b_i)^p, \text{ and} \\ \mathfrak{L}_D^p(\vec{\omega}) &= \sum_{1 \leq j_1 < j_2 \leq q} m_{j_1} m_{j_2} d(a_{j_1}, a_{j_2})^p + \sum_{1 \leq i_1 < i_2 \leq t} n_{i_1} n_{i_2} d(b_{i_1}, b_{i_2})^p,\end{aligned}$$

then we see that $\gamma_D^p(\vec{\omega}) = \mathfrak{R}_D^p(\vec{\omega}) - \mathfrak{L}_D^p(\vec{\omega})$ is the right hand side of the generalized roundness p inequality (8) for the normalized (q,t)-simplex $D(\vec{\omega})$ in X subtract the left hand side of the same inequality. So, by Theorem 6.8, (X,d) has strict p-negative type if and only if $\gamma_D^p(\vec{\omega}) > 0$ for each normalized (q,t)-simplex $D(\vec{\omega})$ in X.

Definition 6.10. Let $p \geq 0$. Let (X,d) be a metric space with p-negative type. We define the *p-negative type gap* of (X,d) to be the non negative quantity

$$\Gamma_X^p = \inf_{D(\vec{\omega})} \gamma_D^p(\vec{\omega})$$

where the infimum is taken over all normalized (q,t)-simplexes $D(\vec{\omega})$ in X.

Notice that if the p-negative type gap $\Gamma_X^p > 0$, then (X,d) has strict p-negative type. Doust and Weston [21] have given an example of an infinite metric space to show that the converse of this statement is not true in general. In other words, there exist infinite metric spaces (X,d) with strict p-negative type and with $\Gamma_X^p = 0$. It is not at all clear that this same phenomenon can occur for finite metric spaces that have strict p-negative type. We will return to this point shortly.

As noted above, finite metric trees have strict 1-negative type. So, it makes sense to try to compute the 1-negative type gap of any given finite metric tree. This has been done recently. Before stating this result, a modicum of notation is necessary. The set of all edges in a metric tree (T,d), considered as unordered pairs, will be denoted $E(T)$, and the metric length $d(x,y)$ of any given edge $e = (x,y) \in E(T)$ will be denoted $|e|$.

Theorem 6.11 (Doust and Weston [21])**.** *Let (T,d) be a finite metric tree with edge set $E(T)$. Then, the 1-negative type gap Γ_T^1 of (T,d) is given by the following formula:*

$$\Gamma_T^1 = \left\{ \sum_{e \in E(T)} |e|^{-1} \right\}^{-1}.$$

In particular, $\Gamma_T^1 > 0$.

In other words, Theorem 6.11 enhances the classical strict 1-negative type inequalities for finite metric trees, via the insertion of a maximal positive additive constant Γ on the left hand side of such inequalities. This is really just a restatement of Theorem 6.11:

Theorem 6.12 (Doust and Weston [21])**.** *Let (T,d) be a finite metric tree. Then for all natural numbers $n \geq 2$, all finite subsets $\{x_1, \ldots, x_n\} \subseteq T$, and all choices of real numbers $\eta_1, \ldots, \eta_n$ with $\eta_1 + \cdots + \eta_n = 0$ and $(\eta_1, \ldots, \eta_n) \neq \vec{0}$, we have*

$$\Gamma + \sum_{1 \leq i,j \leq n} d(x_i, x_j)\eta_i \eta_j \leq 0 \tag{10}$$

where $\Gamma = \Gamma_T^1 = \left\{ \sum\limits_{e \in E(T)} |e|^{-1} \right\}^{-1} > 0$. *Moreover, Γ is the maximal such constant.*

One application of the p-negative type gap Γ_X^p is to exploit it as a device to improve lower bounds on the (supremal) strict negative type of certain finite metric spaces. The idea is as follows.

Theorem 6.13 (Doust and Weston [21])**.** *Let $q \geq 0$. Let (X,d) be a finite metric space with $|X| \geq 3$. If (X,d) has a positive q-negative type gap $\Gamma_X^q > 0$, then there exists $\zeta > 0$ such that (X,d) has strict p-negative type for all $p \in (q-\zeta, q+\zeta)$. Moreover, ζ may be chosen so that it depends only upon Γ_X^q and the set of non zero distances in (X,d).*[7]

Theorems 6.11 and 6.13 generalize the result of Hjorth *et al* [34] on finite metric trees (as discussed in the opening stanza of this section). Namely:

Theorem 6.14 (Doust and Weston [21])**.** *Let (T,d) be a finite metric tree with $|T| \geq 3$. Then there exists an $\zeta > 0$ such that (T,d) has strict p-negative type for all $p \in (1-\zeta, 1+\zeta)$. Moreover, ζ may be chosen so that it depends only upon the unordered distribution of the tree's edge lengths.*

By way of an illustration of this theorem; if (T,d) is a finite metric tree, with say $|T| = n$, where d is the ordinary path metric on T (so that $|e| = 1$ for all edges $e \in E(T)$), then the maximal (strict) p-negative type of (T,d) is at least:

$$1 + \frac{\ln\left(1 + \frac{1}{(n-1)^3(n-2)}\right)}{\ln(n-1)}.$$

This is unlikely to be the optimal such lower bound since the estimates obtained in Doust and Weston [21] only take into account the longest and the shortest non trivial geodesics in such trees. Intermediate distances are not seen to play a role in their arguments surrounding Theorem 6.14.

We conclude this section by compiling a list of open problems concerning strict p-negative type.

Questions. Assume $p > 0$ throughout.

(1) If a metric space (X,d) has (strict) p-negative type, must it have strict q-negative type for some or all positive $q < p$?
(2) Finite subsets X of the sphere $\mathbb{S}^n$, endowed with the natural inherited geodesic metric d, which do not contain more than one pair of antipodal points are know to have strict 1-negative type (Hjorth *et al* [34]). Determine the 1-negative type gap Γ_X^1 of (X,d). Is it necessarily positive? One might reasonably expect that $\Gamma_X^1 \approx 0$ (may be made arbitrarily small) if X *nearly* contains two pairs of antipodal points.

[7]In the case $q = 0$ one must naturally work with the interval $p \in [0, \zeta)$.

(3) Is it possible for a finite metric space (X, d) to have strict p-negative type with a trivial p-negative type gap $\Gamma_X^p = 0$? (As we remarked earlier in this section this is possible for infinite metric spaces. However, the same phenomenon might not persist for finite metric spaces.) A negative answer to this question would imply a negative answer to our next question (on account of Theorem 6.13), and hence *positive* answers to question (2) as above.

(4) Can the maximal p-negative type of a finite metric space be strict? This seems like an important question. If the answer is NO (for example) then *every* finite metric space that has strict 1-negative type would have to have q-negative type for some $q > 1$. The paper of Doust and Weston [21] illustrates that this is the case for finite metric trees. A *negative* answer to this question implies *positive* answers to question (2) above.

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