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UNIFORM ESTIMATE OF RELIABILITY FOR A COMPLEX REGENERATED SYSTEM WITH UNLIMITED NUMBER OF REPAIRS UNITS

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Under minimal (essentially sufficient) conditions the limit theorems are proved, giving an asymptotic estimate of reliability of a complex regenerated system with unlimited number of repairs units.

1. INTRODUCTION

An asymptotical estimate of reliability of a complex regenerated system was found in [1] (its description see below). It was supposed that the distributions of repairs times are fixed, and all intensities of elements rejections are proportional to a small parameter tending to zero. For such a parametric formulation it was shown that the normed by its mean time of faultless work of the system converges weakly to an exponentially distributed value. Besides, and this is the hardest part of the work in the paper we give an asymptotic estimate of the mean time of faultless work. We assume that all distributions of the repairs times have finite moments whose order is equal to the system's excessiveness (i.e., the minimal number of elements whose rejection leads to the rejection of the system). In [2] for very simpler reservation model with reconstruction and unlimited number of repairs units the minimal conditions (convergence by Khinchin) were found under which the same estimate for the distribution of the time of faultless work is valid as in [1].

The aim of this work is the asymptotical analysis of the model of a complex regenerated system with unlimited number of repairs units. We will find the minimal conditions for the distributions, and the limit theorem will be proved in the uniform form when all parameters and distributions defining the model are changed.

2. THE SYSTEM DESCRIPTION

The system consists of n elements each of them can be in good or in bad repair. The state of all elements of the system in each time moment is described by the binary vector $e(t) = [e_1(t),\ e_2(t),\ \ldots,\ e_n(t)]$, where $e_i(t) = 0$ if the ith element is at the moment t in good repair and $e_i(t) = 1$ in the opposite case. Assume that the system's state (it works or it doesn't) is defined uniquely by the states of its elements. Let E_+ be the set of good states of the system and E_- is the set of bad ones. Assume that $\bar{0} = (0, 0, \ldots, 0) \in E_+$, $\bar{1} = (1, 1, \ldots, 1) \in E_-$ and the partition of the whole set of binary vectors: $E = \{e\} = E_+ \bigcup E_-$, $E_+ \bigcap E_- = \varnothing$ is a monotone structure (see [3]). Let $\lambda_i(e)$ be the intensity of the rejection (see [2]) of the ith element in the state e, and $G_i(t) = P\{\eta_i < t\}$ be the distribution function for the time of repairs of the ith element. We consider the case when the number of the repairs units is unlimited, i.e., each element is reconstructed immediately after its rejection. Each element returns after its repairs on its place in the system. At the initial moment $e(0) = \bar{0}$.

The system's behavior is described by a random process $e(t)$ which is under our assumptions regenerating of special type (see [2]). The intervals, where $e(t) \equiv 0$, are called free periods and have the exponential distribution with the parameter

$$\lambda(\bar{0}) = \sum_{i=1}^{n} \lambda_i(0),$$

and the intervals, where $e(t) > 0$, are called busy periods and have the complex distribution with the distribution function $\Phi_0(t)$. All free periods and busy periods are independent in whole. Denote

$$\overline{\lambda} = \max_{e \in E_+} \sum_{i=1}^{n} \lambda_i(e);$$

we assume that $\lambda_i(e) = 0$ if in the state e the ith element rejects.

Denote

$$G(t) = \min_{i=\overline{1,n}} G_i(t), \quad T = \int_0^\infty t dG(t), \quad 1 - G(t) = \overline{G}(t).$$

Let q be the probability of the system rejection at one work period. In [1] the following result was obtained.

If $\overline{\lambda}T \to 0$, *then*

$$P\{\lambda(\overline{0})q\tau > x\} \to e^{-x}, \tag{1}$$

where τ *is the moment of the first system's rejection.*

The main aim of our work is to estimate under minimal conditions the probability q. For this we introduce more notions and notations.

We mean by a way $\pi = \{\overline{0}, e^{(1)}, e^{(2)}, \dots, e^{(m+1)}\}$ the sequence of states passed by the process $e(t)$ from the start of the work till the first rejection at this period. $e^{(k)} \in E_+$, $k = \overline{1,m}$, $e^{(m+1)} \in E_-$. The class of all ways is denoted Π.

The way π_0 is said to be monotone if $e^{(1)} < e^{(2)} < \dots < e^{(m+1)}$; Π_0 is the class of all monotone ways. $\Pi_1 = \Pi \backslash \Pi_0$ is the class of nonmonotone ways. To each monotone way it corresponds the sequence of the numbers of rejecting elements: $\pi_0 \to \{i_1, i_2, \dots, i_m, i_{m+1}\}$.

Let $q(\pi)$ be the probability of the fact that at the work period we will have a rejection of the system and the way π will be passed. Then

$$q = \sum_{\pi \in \Pi} q(\pi) = \sum_{\pi_0 \in \Pi_0} q(\pi_0) + \sum_{\pi_1 \in \Pi_1} q(\pi_1) = q_0 + q_1.$$

The probability of the rejection in the monotone way q_0 is expressed explicitly by the finite sum of integrals

$$q_0 = \sum_{\pi_0 \in \Pi_0} q(\pi_0) = \sum_{\pi_0 \in \Pi_0} \frac{\prod_{k=1}^{m+1} \lambda_{i_k}(e^{(i_{k-1})})}{\lambda(\overline{0})} \int \cdots \int_{0 < t_m < t_{m-1} < \dots < t_1} e^{-s(\pi_0)} \prod_{k=1}^{m} \overline{G}_{i_k}(t_k) dt_1 \dots dt_m, \tag{2}$$

where

$$i_0 = 0, \quad s(\pi_0) = \lambda_{i_1}(e^{(i_1)})(t_1 - t_2) + \dots \lambda_{i_m}(e^{(i_m)})t_m.$$

Denote by $\bar{q}(\pi_0)$ the sum (2) where the factor $e^{-s(\pi_0)}$ is substituted by 1 and let $\bar{q}_0 = \sum_{\pi_0 \in \Pi_0} \bar{q}(\pi_0)$. Our problem is to find the minimal conditions when $q \sim \bar{q}_0$.

For any way π we call essential the rejections which were on the interval from the start of the busy period till the first rejection on it and were not reconstructed at the time moment of this first rejection. Let $i_1, i_2, \dots, i_m, i_{m+1}$ be the numbers of elements in these essential rejections, i_{m+1} be the number of the element, whose rejection led to the system rejection. To each monotone way π_0, on which the elements with the numbers $i_1, i_2, \dots, i_m, i_{m+1}$ rejected, it corresponds a class of nonmonotone ways $\Pi_1(\pi_0)$ with these numbers of essential rejections.

3. MAIN STATEMENTS

Let, besides,

$$\overline{\lambda}_i = \max_{e \in E_+} \lambda_i(e), \quad \underline{\lambda}_i = \min_{\substack{e \in E \\ e_i = 0}} \lambda_i(e), \quad \overline{\lambda} = \max_{e \in E_+} \lambda(e), \quad T_i = \int_0^\infty \overline{G}_i(t) dt.$$

Lemma 1. *If*

(a) $\frac{\bar\lambda_i}{\lambda_i} \leqslant c, \quad i = \overline{1, n};$

(b) $\frac{1}{T_i}\int\limits_0^\infty (1 - e^{-\bar\lambda t})\overline{G}_i(t)dt \to 0, \quad i = \overline{1, n},$ (3)

then $q_0 \sim \bar q_0$.

We omit the proof of this lemma since it is close to the corresponding proof in [1].

Remark. We say that the sequence of random nonnegative values ξ_n tends to zero by Khinchin and write it as $\xi_n \xrightarrow{x} 0$ if

$$\frac{1}{M\xi_n}\int\limits_0^\infty (1 - e^{-x})P\{\xi_n > x\}dx \to 0.$$

Therefore the condition in the lemma means that $\bar\lambda\eta_i \xrightarrow{x} 0$.

We have introduced the class of nonmonotone ways $\Pi_1(\pi_0)$. Divide it in two subclasses: $\Pi_1'(\pi_0)$ contains the ways in which the essential rejection appears at the start of the busy period. $\Pi_1''(\pi_0) = \Pi_1(\pi_0)\setminus = \Pi_1'(\pi_0)$. We denote, respectively:

$$q_1(\pi_0) = \sum_{\pi\in\Pi_1(\pi_0)} q(\pi), \quad q_1'(\pi_0) = \sum_{\pi\in\pi_1'(\pi_0)} q(\pi),$$
$$q_1''(\pi_0) = \sum_{\pi\in\Pi_1''(\pi_0)} q(\pi), \quad q_1(\pi_0) = q_1'(\pi_0) + q''(\pi_0).$$

The probability of the rejection by the monotone way is

$$q_1 = \sum_{\pi_0\in\Pi_0} q_1(\pi_0).$$

Lemma 2. *If the condition* (3) *is fulfilled, then for any monotone way* π_0

$$q_1'(\pi_0) = o[\overline{q}(\pi_0)].$$

Proof. As above we assume that the numbers of rejecting on the way π_0 elements are $1, 2, \dots, m, m+1$. The probability $q_1'(\pi_0)$ is less than the probability of the following event: at the moments $u_0 = 0, u_1, u_2, \dots, u_m$ the elements with the numbers $1, 2, \dots, m, m+1$ rejected and none of them reconstructed till the moment u_m and on the interval $(0, u_m)$ we had at least one additional rejection.

We increase only the probability of this event if we substitute the intensities $\lambda_1(0), \dots, \lambda_m(e^{m-1}), \lambda_{m+1}(e^m)$ by $\bar\lambda_1, \dots, \bar\lambda_{m+1}$. Then after the substitution $u_m - u_{k-1} = t_k$ we have

$$q_1'(\pi_0) \leqslant \frac{1}{\lambda(0)}\bar\lambda_1 \dots \bar\lambda_m\bar\lambda_{m+1} \int\limits_{\Delta_m}\cdots\int (1 - e^{-\bar\lambda t_m})\overline{G}_1(t_1)\dots\overline{G}_m(t_m)dt_1\dots dt_m.$$

Estimating the ratio as in Lemma 1 we get

$$\frac{q_1'(\pi_0)}{\overline{q}(\pi_0)} \leqslant c^{m+1}\frac{\int\limits_{\Delta_m}\cdots\int(1 - e^{-\bar\lambda t_m})\prod\limits_{i=1}^m \overline{G}_i(t_i)dt_i}{\int\limits_{\Delta_m}\cdots\int\prod\limits_{i=1}^m \overline{G}_i(t_i)dt_i} \leqslant c^{m+1}\sum_{i=1}^m \frac{1}{T_i}\int\limits_0^\infty (1 - e^{-\bar\lambda t})\overline{G}_i(t)dt \to 0.$$

The lemma is proved.

Lemma 3. *If* $\bar\lambda T \to 0$ *and* $\frac{\bar\lambda_i}{\lambda_i} \leqslant c$, $i = \overline{1, n}$, *then* $q_1''(\pi_0) = o[\overline{q}(\pi_0)]$.

Proof. Let as in Lemma 2, $0 < u_1 < u_2 < \dots < u_{m+1}$ be the moments of essential rejections of the elements with the numbers $1, 2, \dots, m, m+1$. The probability $q_1''(\pi_0)$ is less than the probability of the following event: the busy period of nonessential rejections is not finished till the moment u_1 and at the moments $u_1, u_2, \dots, u_{m+1}$ we had the rejections of the elements with the numbers $1, 2, \dots, m+1$. This probability does not exceed the value

$$\prod_{i=1}^{m} \lambda_i \int_{0<u_1<\dots<u_{m+1}} \cdots \int \overline{\Phi}(u_1)\overline{G}_1(u_{m+1}-u_1)\dots\overline{G}_m(u_{m+1}-u_m)du_1\dots du_{m+1}.$$

The probability that the busy period of nonessential rejections is not finished to the moment u_1, $\overline{\Phi}(u_1)$, can only increase if we substitute $\lambda_i(e)$ by $\overline{\lambda}_i$ and will assume that every rejected element is substituted immediately by a reserve one. Then each element generates a Poisson flow of rejections with the parameter $\overline{\lambda}_i$. In this case the busy period will not depend to the numbers of essential rejections, we will denote its distribution function by $\Phi_*(u)$

$$\overline{\Phi}(u_1) \leqslant \overline{\Phi}_*(u_1).$$

If we substitute in the last integral $t_k = u_{m+1} - u_k$, $z = u_1$, then

$$q_1''(\pi_0) \leqslant \prod_{i=1}^{m+1} \overline{\lambda}_i \int_{\Delta_m} \cdots \int \left(\int_0^\infty \overline{\Phi}_*(z)dz\right) \prod_{i=1}^{m} \overline{G}_i(t_i)dt_i \leqslant c^{n+1} q(\pi_0)\lambda(\overline{0}) \int_0^\infty \overline{\Phi}_*(z)dz.$$

But $\Phi_*(z)$ is a distribution function of the busy period in the system of mass service $M|G|1|\infty$ in which $\lambda = \sum_{i=1}^{n} \overline{\lambda}_i$ is the intensity of the incoming flow, and $G_0(t) = \sum_{i=1}^{n} \frac{\overline{\lambda}_i}{\lambda} G_i(t)$ is the distribution function of the service time.

For such a system the mean length of the busy time is equal to (see [4])

$$\int_0^\infty \overline{\Phi}_*(z)dz = \frac{T_0}{1-\lambda T_0},$$

where $T_0 = \sum_{i=1}^{n} \frac{\overline{\lambda}_i}{\lambda} T_i \leqslant T$. Since by virtue of the lemma $\overline{\lambda}T \to 0$, we have $\lambda T_0 \to 0$ and

$$\frac{q_1''(\pi_0)}{\overline{q}(\pi_0)} \to 0.$$

The proof is complete.

From the proved lemmas follows

Theorem. *If* $\overline{\lambda}_i/\underline{\lambda}_i \leqslant c$ *and* $\frac{1}{T_i}\int_0^\infty (1-e^{-\overline{\lambda}t})\overline{G}_i(t)dt \to 0$, $i = \overline{1, n}$, *then*

$$\lim P\{\lambda(\overline{0})\overline{q}_0\tau > x\} = e^{-x}.$$

Proof. From Lemmas 2 and 3 we obtain, summing over all monotone ways,

$$q_1 = o(\overline{q}_0).$$

Then from the conditions of Lemma 2 follows assumption (1) (see [2]). Since by Lemma 1 $q_0 \sim \overline{q}_0$ and the conditions of Lemma 3 imply (1), under the conditions of our lemma we have $q \sim \overline{q}_0$ and

$$\lim P\{\lambda(\overline{0})q\tau > x\} = \lim P\{\lambda(\overline{0})\overline{q}_0\tau > x\} = e^{-x}.$$

The proof is complete.

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