A local limit theorem for random walk maxima with heavy tails

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Abstract

For a random walk with negative mean and heavy–tailed increment distribution F, it is well known that under suitable subexponential assumptions, the distribution π of the maximum has a tail $\pi(x,\infty)$ which is asymptotically proportional to $\int_x^\infty F(y,\infty)\,dy$. We supplement here this by a local result showing that $\pi(x,x+z]$ is asymptotically proportional to $zF(x,\infty)$.

Key words: integrated tail, ladder height, subexponential distribution

Let $X_1, X_2, \ldots$ be i.i.d. with common distribution F with mean $-\infty < m < 0$. Set $m_+ = \int_0^\infty \overline{F}(x)\,dx$ and $\overline{F}_I(x) = \int_x^\infty \overline{F}(y)\,dy$. Let

$$S_n = X_1 + \cdots + X_n, \quad M = \max_{n\geq 0} S_n,$$

and $\pi(dx) = \mathbb{P}(M \in dx)$. Assume that G_- and G_+ are respectively the descending and ascending ladder height distributions, i.e. the distributions of S_{τ_-} and S_{τ_+}, where

$$\tau_- = \inf\{n > 0 : S_n \leq 0\}, \quad \tau_+ = \inf\{n > 0 : S_n > 0\}.$$

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Let μ_- be the mean of S_{τ_-}.

We assume throughout that the right tail $\overline{F}(x) = 1 - F(x) = F(x,\infty)$ is proportional to a tail in the class $\mathcal{S}^*$ introduced by Klüppelberg (1988), which is equivalent to

$$\int_0^x \overline{F}(x-y)\overline{F}(y)\,dy \;\sim\; 2m_+\overline{F}(x), \quad x \to \infty\,. \tag{1}$$

This implies that both $\overline{F}(x)$ and the integrated tail $\overline{F}_I(x)$ are subexponential and (1) is only marginally stronger than any of these assumptions. It is then well–known (e.g. Embrechts and Veraverbeke (1982), Embrechts et al. (1997) or Asmussen (2000) and references therein) that

$$\overline{\pi}(x) \sim \frac{1}{|m|}\overline{F}_I(x), \tag{2}$$

$$\overline{G}_+(x) \sim \frac{1}{|\mu_-|}\overline{F}_I(x),\; x \to \infty\,. \tag{3}$$

The relation (2) is of basic importance in insurance risk and queueing theory, and has been generalized to many models (see Asmussen (2000, Ch. IX) for references to the insurance risk literature; some selected queueing papers are Asmussen and Møller (1999), Resnick and Samorodnitsky (1999), Whitt (2000). In this note, we provide local versions of (2). For π:

Theorem 1. Assume that F is non–lattice satisfying (1). Then for each $z > 0$,

$$\pi(x, x+z] \sim \frac{1}{|m|}z\overline{F}(x). \tag{4}$$

In the lattice case, the same conclusion holds when x, z are restricted to values of the lattice span. Further, there exist constants c_1, c_2 such that for all $x, z \geq 0$

$$\pi(x, x+z] \leq (c_1 + c_2 z)\overline{F}(x) \tag{5}$$

This extends a result of Klüppelberg (1989) which requires, among others, the existence of densities and a strong M/G/1 type condition, that the left tail of F is of the form $F(dy) = ce^{\lambda y}$, $y < 0$. We proceed to the proof.

Lemma 1. Let H_1, H_2 be distributions on $(0,\infty)$ such that for each $z > 0$, $H_1(x, x+z] \sim d_1 z\overline{F}(x)$, $H_2(x, x+z] \sim d_2 z\overline{F}(x)$. Then $(H_1 * H_2)(x, x+z] \sim (d_1 + d_2)z\overline{F}(x)$.

Proof. We use repeatedly that

$$\frac{\overline{F}(x-b)}{\overline{F}(x)} \to 1 \quad \text{uniformly in } b \le b_0 \tag{6}$$

by subexponentiality (see Embrechts et al. (1997, p. 577) or Asmussen (2000, p. 253)). The first application is the relation

$$\ell_2 \int_0^a \overline{F}(x-y)\overline{F}(y)\,dy \;=\; \ell_2 \int_{x-a}^x \overline{F}(x-y)\overline{F}(y)\,dy \;=\; m_+ \tag{7}$$

where ℓ_2 is the operator defined by $\ell_2 g(x,a) = \lim_{a\to\infty} \limsup_{x\to\infty} g(x,a)/\overline{F}(x)$, which follows by dominated convergence. Write $(H_1 * H_2)(x, x+z] = J_1 + J_2 + J_3$ where J_1, J_2, J_3 are the integrals of $H_2(x-y, x-y+z]\,H_1(dy)$ over $(0,a]$, $(a, x+z-a]$, resp. $(x+z-a, x+z]$, $a > z$. Then

$$J_1 \;=\; \int_0^a H_2(x-y, x-y+z]\,H_1(dy) \;\sim\; d_2 z\overline{F}(x)H_1(a), \qquad x\to\infty,$$

by dominated convergence and the assumption on H_2. Similarly,

$$J_3 = \int_0^{a-z} H_1(x-v, x-v+z]\,H_2(dv) + \int_{a-z}^a H_1(x-a+z, x-v+z]\,H_2(dv).$$

It follows that

$$\begin{aligned} d_1 z\overline{F}(x)H_2(a-z) &\sim \int_0^{a-z} H_1(x-v, x-v+z]\,H_2(dv) \\ &\le J_3 \le \int_0^a H_1(x-v, x-v+z]\,H_2(dv) \sim d_1 z\overline{F}(x)H_2(a)\,. \end{aligned}$$

As a can be taken arbitrarily large, it is left to show that $\ell_2 J_2 = 0$, and here it suffices to restrict a to integer values. But letting $x^* = \lceil x+z \rceil$, it follows by repeatedly using (6) and the assumptions on H_1, H_2 that we can bound $\ell_2 J_2$ by

$$d_2 z\,\ell_2 \int_a^{x^*-a} \overline{F}(x^*-y)\,H_1(dy)$$

$$= d_2 z\, \ell_2 \sum_{k=a}^{x^*-a-1} \overline{F}(x^* - k - 1) H_1(k, k+1]$$

$$= d_1 d_2 z\, \ell_2 \sum_{k=a}^{x^*-a-1} \overline{F}(x^* - k - 1) \overline{F}(k)$$

$$= d_1 d_2 z\, \ell_2 \int_a^{x^*-a} \overline{F}(x^* - y) \overline{F}(y)\, dy\,,$$

which is 0 according to (1) and (7). □

Lemma 2. Let H be a distribution on $(0, \infty)$ such that for each $z > 0$, $H(x, x+z] \sim dz\overline{F}(x)$. Then for each $\delta > 0$ there is $C_\delta < \infty$ such that $\alpha_n \le C_\delta (1+\delta)^n$ where $\alpha_n = \sup_{x>0} H^{*n}(x, x+z]/\overline{F}(x)$.

Proof. We start by choosing a such that

$$\int_a^{x+z-a} \overline{F}(x-y) H(dy) \;\le\; \frac{\delta}{2} \overline{F}(x) \tag{8}$$

for all x.

The existence of a follows since the proof of Lemma 1 shows that (8) can be obtained for all large x, say $x \ge x_1$, and the validity for all x then follows by replacing a by a larger a if necessary, say $(x_1+z)/2$ (making the integral non-positive for $x \le x_1$).

Next we choose x_0 such that $\overline{F}(x-a)/\overline{F}(x) \le 1+\delta/2$ for all $x \ge x_0$. We write $H^{*n}(x, x+z] = J_1(n,x) + J_2(n,x) + J_3(n,x)$ where $J_1(n,x)$, $J_2(n,x)$, $J_3(n,x)$ are the integrals over $[0,a]$, $(a, x+z-a]$, resp. $(x+z-a, x+z]$, of $H^{*(n-1)}(x-y, x+z-y]$ w.r.t. $H(dy)$.

To bound $J_1(n+1,x)$, we first note that for some $D_1 < \infty$ and some $\eta < 1$ it holds that $H^{*n}(x_0+a+z) \le D_1 \eta^n$, cf. Asmussen (1987, p. 113). Hence for $x \le x_0 + a$,

$$\begin{aligned} J_1(n+1,x) \;&\le\; \int_0^a H^{*n}(x+z-y)\, H(dy) \;\le\; H^{*n}(x+z) H(a) \le \\ &\le\; H^{*n}(x_0+a+z) \;\le\; D_1 \eta^n \;\le\; D_2 \eta^n \overline{F}(x), \end{aligned}$$

with $D_2 = D_1/\overline{F}(x_0+a)$, whereas for $x > x_0 + a$

$$J_1(n+1,x) \;\le\; \alpha_n \int_0^a \overline{F}(x-y)\, H(dy) \;\le\; \alpha_n \overline{F}(x-a)H(a) \;\le\; \alpha_n(1+\delta/2)\overline{F}(x).$$

All together we have

$$J_1(n+1,x) \;\le\; [D_2\eta^n + \alpha_n(1+\delta/2)]\overline{F}(x). \tag{9}$$

Further, by (8)

$$J_2(n+1,x) \;\le\; \alpha_n \int_a^{x+z-a} \overline{F}(x-y)\, H(dy) \;\le\; \overline{F}(x)\alpha_n\delta/2.$$

For $J_3(n+1,x)$, we have as in the proof of Lemma 1 that $J_3(n+1,x) \le \int_0^a H(x-y, x+z-y)\, H^{*n}(dy)$. For $x \le x_0 + a$, this yields the bound

$$J_3(n+1,x) \;\le\; H^{*n}(a) \le D_3\eta^n \;\le\; D_4\eta^n\overline{F}(x),$$

with $D_4 = D_3/\overline{F}(x_0+a)$, whereas for $x > x_0 + a$ we get the bound

$$J_3(n+1,x) \;\le\; \alpha_1 \int_0^a \overline{F}(x-y)H^{*n}(dy) \;\le\; \alpha_1(1+\delta/2)\overline{F}(x)H^{*n}(a)$$

so that all together

$$J_3(n+1,x) \;\le\; [D_4\eta^n + \alpha_1(1+\delta/2)H^{*n}(a)]\overline{F}(x). \tag{10}$$

Adding the obtained bounds for the $J_i(n+1,x)$ and dividing by $\overline{F}(x)$ yields

$$\alpha_{n+1} \;\le\; \alpha_n(1+\delta) + D,$$

with $D = (D_2 + D_4)\eta^n + \alpha_1(1+\delta/2)H^{*n}(a)$. Now according to the treatment of (c) of Lemma 1.3.5 in Embrechts et al. (1997), we take that: for each $\delta > 0$ there exists some $0 \le D = D(\delta) < \infty$ such that, uniformly for each integer n

$$\alpha_{n+1} \le \alpha_n(1+\delta) + D, \tag{11}$$

then, by simple recursion treatment we obtain $\alpha_n \le (\alpha_1 + D/\delta)(1+\delta)^n$ and therefore Lemma 2 is proved.

□

Let $\|H\|$ denote the total mass of a measure H.

Lemma 3. For each $n \geq 1$, $G_+^{*n}(x, x+z] \;\sim\; \dfrac{n\|G_+\|^{n-1}}{|\mu_-|} z\overline{F}(x)$.

Proof. We consider only the non–lattice case, the lattice case being entirely similar. With $U_- = \sum_0^\infty G_-^{*n}$, it is standard (e.g. the proof of (2) in Asmussen (2000, Sect. IX.3)) that

$$
\begin{aligned}
G_+(x, x+z] &= \int_{-\infty}^{0} F(x-y, x+z-y]\, U_-(dy) \\
&= \int_{x}^{\infty} U_-(x-v, x+z-v]\, F(dv). \qquad (12)
\end{aligned}
$$

By Blackwell's renewal theorem, $U_-(-w, -w+z]$ has limit $z/|\mu_-|$ as $w \to \infty$ and is bounded by $c_1 + c_2 z$ uniformly in w. Since $\overline{F}(x+a)/\overline{F}(x) \to 1$ for each a, it follows by letting first $x \to \infty$ and next $a \to \infty$ in the inequality

$$
\begin{aligned}
\frac{G_+(x, x+z]}{\overline{F}(x)} &\leq \sup_{w \geq a} U_-(-w, -w+z] \frac{\overline{F}(x+a)}{\overline{F}(x)} \\
&\quad + (c_1 + c_2 z) \frac{\overline{F}(x) - \overline{F}(x+a)}{\overline{F}(x)}
\end{aligned}
$$

that $\limsup_{x\to\infty} [G_+(x, x+z]/\overline{F}(x)] \leq z/|\mu_-|$. The proof of $\liminf_{x\to\infty} \geq z/|\mu_-|$ is similar and gives the case $n = 1$ of the lemma. The case $n > 1$ follows in a straigtforward way by induction and Lemma 1 applied to $G_+/\|G_+\|$. □

Proof of Theorem 1. As we have mentioned, $U_-(-w, -w+z]$ is bounded by $c_1 + c_2 z$ uniformly in w. Therefore, $G_+(x, x+z] \leq (c_1 + c_2 z)\overline{F}(x)$, so that the assumptions of Lemma 2 hold for $H = G_+/\|G_+\|$. Choosing δ such that $\gamma = (1+\delta)\|G_+\| < 1$, we therefore have

$$
G_+^{*n}(x, x+z] \leq C_\delta \gamma^n \overline{F}(x). \qquad (13)
$$

Using $\pi = (1 - \|G_+\|) \sum_0^\infty G_+^{*n}$ (Asmussen, 2000, formula (3.2) p. 261), dominated convergence and Lemma 3 therefore yields

$$
\begin{aligned}
\frac{\pi(x, x+z]}{\overline{F}(x)} &= (1 - \|G_+\|) \sum_0^\infty \frac{G_+^{*n}(x, x+z]}{\overline{F}(x)} \\
&\to (1 - \|G_+\|) \frac{z}{|\mu_-|} \sum_0^\infty n\|G_+\|^{n-1} \;=\; \frac{z}{|\mu_-|(1 - \|G_+\|)}.
\end{aligned}
$$

Now just note that $m = (1-\|G_+\|)\mu_-$ (see again Asmussen (2000, Sect. IX.3)).

The inequality (5) follows from (13). □

One application of Theorem 1 is

Corollary 1. Let $g(t)$ be directly Riemann integrable on $[0, \infty)$. Then

$$\int_x^\infty g(y-x)\,\pi(dy) \;\sim\; \frac{1}{|m|}\overline{F}(x)\int_0^\infty g(t)\,dt \tag{14}$$

This follows by using Theorem 1 precisely as Blackwell's renewal theorem is used in the proof of the key renewal theorem (Asmussen (1987, p. 119)). In Asmussen (1998) and Kalashnikov and Konstantinides (2000), an estimate of the type (14) was stated to be an easy consequence of (2); however, a supporting argument is required.

For the sake of completeness, we finally mention that one also easily obtains a GI/G/1 version of the M/G/1 result of Klüppelberg (1989) on the density of π (the proof is similar or as the proof of (2) in Asmussen (2000)):

Proposition 1. Assume instead of (1) that F has a density $f(x)$ for $x > 0$ such that the function $f_+(x)$ defined by $f_+(x) = f(x)/\overline{F}(0)$, $x > 0$, $f_+(x) = 0$, $x \le 0$, satisfies $f_+^{*n}(x) \sim nf_+(x)$ for all n and that to each $\epsilon > 0$ there exists c_ϵ such that $f_+^{*n}(x) \le c_\epsilon(1+\epsilon)^n f_+(x)$ for all n and x. Then π has a density $g(x)$ satisfying $g(x) \sim f(x)/|m|$.

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