EXTENSIONS OF COMPACTNESS OF TYCHONOFF POWERS OF 2 IN ZF

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NOTATION AND TERMINOLOGY

Let (X, T) be a topological space. There are various definitions of compactness for topological spaces which are equivalent in **ZFC** (Zermelo-Fraenkel set theory + Axiom of Choice **AC**), but split in **ZF** (Zermelo-Fraenkel set theory minus **AC**). Here, we use the **Heine-Borel** definition of compactness, that is,

X is **compact** if every open cover of X has a finite subcover.

X is **countably compact** if every countable open cover of X has a finite subcover.

X is **Lindelöf** if every open cover of X has a countable subcover.

Let X be a non empty set.

2^X denotes the Tychonoff product of the discrete space $2 = \{0, 1\}$ and,

$$\mathcal{B}_X = \{[p] : p \in \mathsf{Fn}(X, 2)\},$$

where $\mathsf{Fn}(X, 2)$ is the set of all finite partial functions from X into 2 and

$$[p] = \{f \in 2^X : p \subset f\},$$

will denote the standard (clopen) base for the topology on 2^X. For every $n \in \mathbb{N}$, let

$$\mathcal{B}_X^n = \{[p] \in \mathcal{B}_X : |p| = n\}.$$

We call the elements of $\mathcal{B}_X^n$, $n \in \mathbb{N}$, n-**basic open sets** of 2^X. Clearly,

$$\mathcal{B}_X = \cup\{\mathcal{B}_X^n : n \in \mathbb{N}\}.$$

For $n \in \mathbb{N}$, 2^X is **compact-n** if every cover $\mathcal{U} \subset \mathcal{B}_X^n$ of 2^X has a finite subcover.

TP(2^X) : 2^X is compact.

TPC(2^X) : 2^X is countably compact.

BPI (Boolean Prime Ideal Theorem) : Every Boolean algebra has a prime ideal.

UF(ω) : There exists a free ultrafilter on ω.

CAC : AC restricted to countable families of non empty sets.

CAC($\mathbb{R}$) : CAC restricted to countable families of non empty sets of reals.

KNOWN RESULTS

1. J. Mycielsky (1964) : BPI iff $\forall\, X$, 2^X is compact.

2. J. Truss (1984): Use of compactness of $2^{\mathbb{R}}$ in his research.

3. P. Howard and J. E. Rubin (1998) : Is the statement ``$2^{\mathbb{R}}$ is compact" provable in ZF?

4. K. Keremedis (2000) : It is relatively consistent with ZF that $2^{\mathbb{R}}$ is **not** compact.

5. K. Keremedis (2005) : CAC, hence CAC($\mathbb{R}$), does not imply TP($2^{\mathbb{R}}$) in ZF.

6. K. Keremedis, E. Felouzis, E. Tachtsis (2007) : It is relatively consistent with ZF that $2^{\mathbb{R}}$ is **not** countably compact.

PROBLEMS

In the paper:

K. Keremedis, E. Felouzis, E. Tachtsis, *On the compactness and countable compactness of* $2^{\mathbb{R}}$ *in ZF,* Bull. Polish Acad. Sci. Math. **55** (2007), 293-302

the following questions are asked:

(1) **Does CAC($\mathbb{R}$) imply TPC($2^{\mathbb{R}}$) in ZF**?

(2) **Does TPC($2^{\mathbb{R}}$) imply TP($2^{\mathbb{R}}$) in ZF**?

The above problems are the motivation of this paper.

In this paper, we shall show that:

(1) BPI iff for every set X and for every $n \in \mathbb{N}\backslash\{1\}$, 2^X is countably compact and compact-n.

In particular, TP($2^{\mathbb{R}}$) iff TPC($2^{\mathbb{R}}$) + ``$2^{\mathbb{R}}$ is compact-n for every $n \in \mathbb{N}\backslash\{1\}$".

(2) CAC + UF(ω) implies ($\forall X$) TPC(2^X).

(3) CAC, hence CAC($\mathbb{R}$), does not imply ``for all integers $n > 1$, $2^{\mathbb{R}}$ is compact-n" in ZF.

(4) It is not provable in ZF that for every infinite set X and for every $n \in \mathbb{N}$, TPC(2^X) implies 2^X is compact-n. In particular, it is not provable in ZF that for every infinite set X, TPC(2^X) implies TP(2^X).

MAIN RESULTS

Theorem 1 *Let X be a set and let $n \in \mathbb{N}\backslash\{1\}$. If 2^X is compact-n, then every disjoint family of n-element subsets of X has a choice function. In particular, the statement "$\forall\, X, \forall\, n \in \mathbb{N}$, 2^X is compact-n" is not a theorem of ZF.*

Proof. Fix a disjoint family $\mathcal{A} = \{A_i : i \in I\}$ of n-element subsets of X. By way of contradiction assume that $\mathcal{A}$ has no choice function. Then

$$\mathcal{U} = \{[p] : \exists i \in I,\, p \in 2^{A_i},\, (p \equiv 0 \vee |p^{-1}(1)| \geq 2)\}$$

is an n-basic open cover of 2^X which clearly has no finite subcover. This contradicts the fact that 2^X is compact-n.

For the second assertion, we use **Cohen's second forcing model** in which there is a countable disjoint family of 2-element subsets of $\mathcal{P}(\mathbb{R})$ having no choice function in the model. Thus, $2^{\mathcal{P}(\mathbb{R})}$ fails to be compact-2 in this model. ■

Remark 1 (1) **It is provable in ZF that for every set X, 2^X is compact-1**. (Let $\mathcal{U}$ be a 1-basic open cover of 2^X. If $\mathcal{U}$ has no finite subcover, then for every $x \in X$, the family

$$\mathcal{U}_x = \{O : O \text{ is open in } 2 \text{ and } \pi_x^{-1}(O) \in \mathcal{U}\}$$

is not a cover of 2. For each $x \in X$, let u_x be the least element of $2 \setminus (\cup \mathcal{U}_x)$. Then $(u_x)_{x \in X} \notin \cup \mathcal{U}$, a contradiction.)

(2) **The implication of the previous theorem is not reversible in ZF**. In particular, it is relatively consistent with ZF that there exists a set X such that for all $n \in \mathbb{N}$, every disjoint family of n-element subsets of X has a choice function, while 2^X fails to be compact-n.

Lemma 1 *Let $n \in \mathbb{N}\backslash\{1\}$. If 2^X is compact-n, then 2^X is compact-m for all integers $m < n$.*

Proof. Fix an integer $m < n$ and let $\mathcal{V} \subset \mathcal{B}_X^m$ be a cover of 2^X. Put

$$\mathcal{W} = \{W \in \mathcal{B}_X^n : \exists\, V \in \mathcal{V},\ W \subset V\}.$$

Clearly, $\mathcal{W}$ is a cover of 2^X, hence by hypothesis, it has a finite subcover, say $\{W_{i_1}, W_{i_2}, \ldots, W_{i_k}\}$ for some $k \in \mathbb{N}$.

For each $j \leq k$ choose $V_{i_j} \in \mathcal{V}$ s/t $W_{i_j} \subset V_{i_j}$.

Then $\{V_{i_1}, V_{i_2}, \ldots, V_{i_k}\}$ is a finite subcover of $\mathcal{V}$, finishing the proof of the lemma. ■

Theorem 2 *The following are equivalent in ZF:*

(1) BPI.

(2) $\forall\, X$, $\forall\, n \in \mathbb{N}$, 2^X *is countably compact and compact-n.*

Proof. It suffices to show that (2) $\rightarrow$ (1). Invoking Mycielski's result we show that for every set X, 2^X is compact, so let X be any (infinite) set and let $\mathcal{U}$ be a basic open cover of 2^X.

For each $n \in \mathbb{N}$, let

$$O_n = \cup\{[p] \in \mathcal{U} : |p| = n\}.$$

Clearly, $\mathcal{O} = \{O_n : n \in \mathbb{N}\}$ is an open cover of 2^X and since 2^X is countably compact, $\mathcal{O}$ has a finite subcover, say $\mathcal{Q} = \{O_{i_1}, O_{i_2}, \ldots, O_{i_r}\}$. Then

$$\mathcal{R} = \{[p] \in \mathcal{U} : \exists\, j \leq r,\ |[p]| = i_j,\ [p] \subset O_{i_j}\}$$

is a cover of 2^X.

Let $i^* = \max\{i_1, i_2, \ldots, i_r\}$.

Since 2^X is compact-i^*, it follows by the previous Lemma that $\mathcal{R}$ has a finite subcover, hence $\mathcal{U}$ has a finite subcover as required. ∎

K. Keremedis, E. Felouzis, E. Tachtsis (2007) :

$$\textbf{BPI} \Leftrightarrow (\forall X, 2^X \textbf{ is Lindelöf}) + \textbf{CAC}_{fin}$$

(CAC_{fin} = *Countable choice for non empty finite sets*).

We improve the above result here by showing that:

Theorem 3 *The following are equivalent in ZF:*

(i) BPI.

(ii) For every set X, 2^X is Lindelöf.

(iii) For every set X, every basic open cover of 2^X has a countable subcover.

Proof. (i) $\rightarrow$ (ii) and (ii) $\rightarrow$ (iii) are straightforward.

(iii) $\rightarrow$ (i). It suffices to show that (iii) $\Rightarrow$ CAC$_{fin}$.

To this end, let

$$\mathcal{A} = \{A_i : i \in \omega\}$$

be a disjoint family of non empty finite sets.

(iii) implies that every non countable subset of $2^{\cup\mathcal{A}}$ has a limit point.

Indeed, let G be a non countable subset of $2^{\cup\mathcal{A}}$. Towards a contradiction, suppose that G has no limit points, then G is a closed set. Consider the following collection of basic open subsets of $2^{\cup\mathcal{A}}$:

$$\mathcal{U} = \{[p] \in \mathcal{B}_{2^{\cup\mathcal{A}}} : (|[p] \cap G| = 1) \vee ([p] \subset 2^{\cup\mathcal{A}} \setminus G)\}.$$

Clearly, $\mathcal{U}$ is a cover of $2^{\cup\mathcal{A}}$, hence by our hypothesis, $\mathcal{U}$ has a countable subcover, say $\mathcal{V}$. It can be readily verified that $|G| \leq |\mathcal{W}|$, where

$$\mathcal{W} = \{[p] \in \mathcal{V} : |[p] \cap G| = 1\}.$$

Thus, G is a countable set. This is a contradiction.

For each $i \in \omega$, let

$$B_i = \{f \in 2^{\cup \mathcal{A}} : (\forall\, j \leq i,\ |f^{-1}(1) \cap A_j| = 1)$$

$$\wedge\, (\forall\, j > i,\ A_j \subset f^{-1}(0))\}.$$

Clearly, $|B_i| < \aleph_0$ for all $i \in \omega$. Put

$$B = \cup\{B_i : i \in \omega\}.$$

We consider the following two cases.

(1) $|B| = \aleph_0$. Then fixing an enumeration for B and picking the least element from each B_i with respect to the prescribed enumeration of B, we may easily define a choice function of $\mathcal{A}$.

(2) $|B| \neq \aleph_0$. Then B has a limit point, say g. We assert that $|g^{-1}(1) \cap A_i| = 1$ for all $i \in \omega$. Assuming the contrary, it follows that $|g^{-1}(1) \cap A_{i_0}| \neq 1$ for some $i_0 \in \omega$. There are two cases:

(i) $A_{i_0} \subset g^{-1}(0)$. Then

$$O_g = [\{(x, 0) : x \in A_{i_0}\}]$$

is a neighborhood of g meeting at most $\cup\{B_j : j < i_0\}$ which is a finite set. This is a contradiction since g is a limit point of B, hence every neighborhood of g must meet B in an infinite set ($2^{\cup\mathcal{A}}$ is a Hausdorff space).

(ii) $|g^{-1}(1) \cap A_{i_0}| \geq 2$. Let $x, y \in A_{i_0}$ be such that $g(x) = g(y) = 1$. Consider the neighborhood

$$O_g = [\{(x, 1), (y, 1)\}]$$

of g. By the definition of B, it readily follows that $O_g \cap B = \varnothing$, and we have reached again a contradiction.

From cases (i) and (ii) we infer that for all $i \in \omega$, $|g^{-1}(1) \cap A_i| = 1$ as asserted. Then, $C = g^{-1}(1)$ is a choice set of $\mathcal{A}$. ■

Theorem 4 *In ZF,* ***CAC + UF(ω)*** $\Rightarrow$ ***($\forall X$) TPC(2^X)****, where*
CAC = Countable axiom of choice,
UF(ω) = there exists a free ultrafilter on ω,
TPC(2^X) = 2^X is countably compact.

Proof. Let X be any infinite set. Fix a nested family $\mathcal{G} = \{G_i : i \in \omega\}$ of closed subsets of 2^X. Towards a contradiction, assume that $\bigcap \mathcal{G} = \varnothing$. By CAC, let $C = \{f_n : n \in \omega\}$ be a choice set of $\{G_n \backslash G_{n+1} : n \in \omega\}$. Let, by UF($\omega$), $\mathcal{F}$ be a free ultrafilter on C. Put

$$\mathcal{H} = \{Y \subset 2^X : Y \cap C \in \mathcal{F}\}.$$

Then $\mathcal{H}$ is an ultrafilter on 2^X, and since, **in ZF, every ultrafilter on 2^X converges**, it follows that $\mathcal{H}$ converges to a point $g \in 2^X$. Since $\mathcal{F}$ is free, it follows that for every open neighborhood O_g of g, $O_g \cap C$ is an infinite set. Thus, $g \in \bigcap \mathcal{G}$. This is a contradiction, finishing the proof of the theorem. ■

Theorem 5 *In every permutation model of ZFA, CAC implies* 2^X *is countably compact for every infinite set* X*.*

Proof. In every permutation model, $\mathbb{R}$ is well orderable (being a pure set), hence UF(ω) holds. ■

Theorem 6 *It is not provable in ZF that for every infinite set* X *and for every* $n \in \mathbb{N}$*, TPC*(2^X) *implies* 2^X *is compact-*n*. In particular, it is not provable in ZF that for every infinite set* X*, TPC*(2^X) *implies TP*(2^X)*.*

Proof. We exhibit a permutation model of CAC + UF(ω) + $\neg$ BPI. Similarly to Cohen's second model, the forcing analogue of the permutation model can be constructed having the desired properties.

The set of atoms A has size $\aleph_1$ and A is a disjoint union of $\aleph_1$ pairs $A_i = \{a_i, b_i\}$, $i \in \aleph_1$.

Let G be the group of permutations of A which fix each A_i.

Let F be the (normal) filter generated by the subgroups

$$\text{fix}(E) = \{\pi \in G : \forall e \in E,\ \pi(e) = e\},$$

where E is a countable subset of A. Thus, the normal ideal of supports is the set of all countable subsets of A.

Let $\mathcal{N}$ be the permutation model determined by G and F.

Then the following hold:

(1). The family $B = \{A_i : i \in \aleph_1\}$ has no choice function in $\mathcal{N}$. Therefore, **BPI is false in** $\mathcal{N}$. In particular, 2^A **is not compact-2 in** $\mathcal{N}$.

(2). Due to the countable supports and the regularity of $\aleph_1$, every function on ω with values in $\mathcal{N}$ belongs to $\mathcal{N}$, hence **DC (= principle of dependent choices), and thus CAC, holds in** $\mathcal{N}$.

Since UF(ω) holds in $\mathcal{N}$, it follows, by CAC + UF(ω) in $\mathcal{N}$, that TPC(2^X) holds in $\mathcal{N}$ for every set X. Hence, the independence result. ∎

Remark 2 It is known (Howard, Keremedis, Rubin, Stanley, 2000) that, in ZF, *the Tychonoff product of countably many compact spaces is countably compact iff it is compact.* By Theorem 6, this ceases to be true in ZF if we consider Tychonoff products of **non-countable** families of compact spaces.

Theorem 7 *For every integer $n > 1$, ``$2^{\mathbb{R}}$ is compact-$n$" implies ``every family $\mathcal{A}$ of $\leq n$-element subsets of $\mathcal{P}(\mathbb{R})$ such that $\bigcup \mathcal{A}$ is disjoint, has a choice set".*

In particular, the statement ``for every integer $n > 1$, $2^{\mathbb{R}}$ is compact-n" is not provable in ZF.

Proof. The proof is by induction on n.

For $n = 2$, assume that $2^{\mathbb{R}}$ is compact-2.

Let $\mathcal{A} = \{T_i : i \in I\}$ be a family of 2-element subsets of $\mathcal{P}(\mathbb{R})$ such that $\cup\mathcal{A}$ is disjoint.

By way of contradiction assume that $\mathcal{A}$ has no choice set.

Consider the following collection of 2-basic clopen subsets of $2^{\mathbb{R}}$.

$$\mathcal{U} = \{[p] \in \mathcal{B}^2_{\mathbb{R}} : (\exists a \in 2) \ \wedge \ (\exists i \in I, \, \forall X \in T_i,$$
$$|p^{-1}(a) \cap X| = 1)\}.$$

We assert that $\mathcal{U}$ is a cover of $2^{\mathbb{R}}$.

Indeed, let $f \in 2^{\mathbb{R}}$. If $f \notin \cup\mathcal{U}$, then

for every $i \in I$, f separates the elements of T_i, that is,

$\exists X \in T_i$ such that $f|_X \equiv 0$ and $f|_{(\cup T_i)\setminus X} \equiv 1$.

It follows that $f^{-1}(0)$ (or $f^{-1}(1)$) is a choice set of the family $\mathcal{A}$, contradicting our assumption that $\mathcal{A}$ admits no choice sets.

Therefore, $f \in \cup\mathcal{U}$ and $\mathcal{U}$ is a cover of the Tychonoff product $2^{\mathbb{R}}$.

On the other hand, $\mathcal{U}$ **has no finite subcover**, contradicting the fact that $2^{\mathbb{R}}$ is compact-2. Hence, $\mathcal{A}$ has a choice set as required.

Assume that for all $m < n$, if $2^{\mathbb{R}}$ is compact-m, then every family $\mathcal{A}$ of $\leq m$-element subsets of $\mathcal{P}(\mathbb{R})$ such that $\bigcup \mathcal{A}$ is disjoint, has a choice set.

We show the result under the premise that $2^{\mathbb{R}}$ is compact-n, where $n > 2$. By Lemma 1 we have

(∗) $2^{\mathbb{R}}$ is compact-m for all $m < n$.

Fix a family $\mathcal{A} = \{T_i : i \in I\}$ of $\leq n$-element subsets of $\mathcal{P}(\mathbb{R})$ such that $\bigcup \mathcal{A}$ is disjoint.

In view of (∗), the induction hypothesis, and the fact that $\mathcal{P}(n)$ is finite, we may assume, without loss of generality, that $|T_i| = n$ for all $i \in I$.

By way of contradiction suppose that $\mathcal{A}$ does not have any choice sets.

Consider the following collection of n-basic clopen subsets of $2^{\mathbb{R}}$.

$$\mathcal{U} = \{[p] \in \mathcal{B}^n_{\mathbb{R}} : (\exists a \in 2) \ \wedge \ (\exists i \in I, \, \forall X \in T_i,$$

$$|p^{-1}(a) \cap X| = 1)\}.$$

We assert that $\mathcal{U}$ is a cover of $2^{\mathbb{R}}$. To see this, let $f \in 2^{\mathbb{R}}$. If $f \notin \cup\mathcal{U}$, then

$\forall i \in I, \ \exists \, X, Y \in T_i$ such that $f|_X \equiv 0$, $f|_Y \equiv 1$.

For every $i \in I$, let $S_i = f^{-1}(0) \cap T_i$. Then S_i is a non empty proper subset of T_i and since $\mathcal{A}$ has no choice set, $|S_i| > 1$ for all $i \in I$.

On the other hand, since for all $i \in I$, the set T_i has n elements, it follows that

$\exists m < n$ such that $|S_i| \leq m$ for all $i \in I$.

Let m_0 be the least such m. Since $m_0 < n$, $2^{\mathbb{R}}$ is compact-m_0, hence by (IH) we have that

$$\mathcal{B} = \{S_i : i \in I\},$$

hence $\mathcal{A}$, has a choice set. This contradicts our assumption on $\mathcal{A}$. Thus, $\mathcal{U}$ is a cover of $2^{\mathbb{R}}$.

On the other hand, $\mathcal{U}$ has no finite subcover, contradicting the fact that $2^{\mathbb{R}}$ is compact-n. Thus, $\mathcal{A}$ does have a choice set. The induction terminates as well as the proof of the first assertion of the theorem.

For the second assertion of the theorem, we invoke **Feferman's forcing model**. In this model, the family

$$\mathcal{A} = \{\{[X], [\omega \backslash X]\} : X \subset \omega\},$$

where

$$[X] = \{Y \subset \omega : |X \triangle Y| < \aleph_0\}$$

has no choice function.

Thus, the statement ``$2^{\mathbb{R}}$ is compact-2" is not valid in Feferman's model, and even more (by Lemma 1) $2^{\mathbb{R}}$ fails to be compact-n for every integer
$n > 1$. This completes the proof of the theorem. ■

Theorem 8 *It is not provable in ZF that CAC implies ``for all integers $n > 1$, $2^{\mathbb{R}}$ is compact-n".*

Proof. In Feferman's model, **AC for well orderable families of non empty sets**, hence CAC, holds whereas by the proof of the second assertion of the previous Theorem, $2^{\mathbb{R}}$ fails to be compact-2 in the model. ■

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