Different versions of a first countable space without choice

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Notation and terminology

(1) The Axiom of Choice (**AC**): *For every family $\mathcal{A}$ of non-empty sets there exists a function $f : \mathcal{A} \to \bigcup \mathcal{A}$ such that $f(x) \in x$ for every $x \in \mathcal{A}$.*

(2) **ZFC**: Zermelo-Fraenkel set theory plus **AC**.

(3) **ZF**: Zermelo-Fraenkel set theory minus **AC**.

(4) **ZF**^{0}: Zermelo-Fraenkel set theory minus the Axiom of Regularity.

(5) Let (X, T) be a topological space. X satisfies:

(A) if $(\forall x \in X)(\exists \mathcal{B}(x))|\mathcal{B}(x)| \leq \aleph_0$ and $\mathcal{B}(x)$ is a local base at x,

(B) if $(\exists (\mathcal{B}(x))_{x \in X})(\forall x \in X)|\mathcal{B}(x)| \leq \aleph_0$ and $\mathcal{B}(x)$ is a local base at x,

(C) if $(\exists (\mathcal{B}(n, x))_{n \in \mathbb{N}, x \in X})(\forall x \in X)\{\mathcal{B}(n, x) : n \in \mathbb{N}\}$ is a local base at x.

Introduction

In **ZF** a topological space (X, T) is called **first countable** if and only if X satisfies **(A)**. Furthermore, in **ZFC** the aforementioned three formulations of first countability are equivalent to each other, that is

(ZFC) : **(A)** $\Leftrightarrow$ **(B)** $\Leftrightarrow$ **(C)**.

On the other hand, in **ZF** the following implications hold:

(ZF) : **(C)** $\Rightarrow$ **(B)** $\Rightarrow$ **(A)**.

G. Gutierres in his paper:

What is a first countable space?, Topology and its Applications, **153** (2006), 3420-3429,

established that **(B)** $\Leftrightarrow$ **(C)** is not provable in **ZF** and posed the following question:

Is the implication (A) $\Rightarrow$ (B) provable in ZF?

- In the present paper **we show that the answer to the latter open question is in the negative in the framework of ZF** by constructing a suitable model for **ZF** in which there exists a topological space which satisfies **(A)** and $\neg$**(B)**.

- We also introduce two weak choice principles which are deducible from **(A)** $\Rightarrow$ **(B)** and **(B)** $\Rightarrow$ **(C)**, respectively, and study the interrelation between **(A)** $\Rightarrow$ **(B)** and **(B)** $\Rightarrow$ **(C)**.

Main results

Definition 1 1. A linearly ordered set $(L, \leq)$ is called **conditionally complete** if every non-empty subset of L with an upper bound has a least upper bound.

2. A linearly ordered set $(L, \leq)$ is called **separable** if L with the order topology is separable.

Theorem 1 *(**(A)** $\Rightarrow$ **(B)**) implies the choice principle: For every family $\mathcal{A} = \{(A_i, \leq_i) : i \in I\}$ of separable conditionally complete linearly ordered sets without endpoints, there exists a function f such that for each $i \in I$, $f(i)$ is a non-empty countable subset of A_i.*

Proof. Fix a family $\mathcal{A}$ as in the statement of the theorem. Without loss of generality assume that $\mathcal{A}$ is pairwise disjoint. Let $\{\infty_i : i \in I\}$ be a set of distinct elements which is disjoint from $\bigcup \mathcal{A}$. For every $i \in I$, let $X_i = A_i \cup \{\infty_i\}$. We extend the order of A_i to X_i by declaring that ∞_i is less than every member of A_i.

For each $i \in I$, consider the collection

$$T_i = \{[\infty_i, a) : a \in A_i\} \cup \{\emptyset, X_i\}.$$

T_i is a topology on X_i. It suffices to show that if $M \subset A_i$, then $U_M = \bigcup\{[\infty_i, a) : a \in M\} \in T_i$, since the other two requirements for T_i to be a topology are straightforward. There are two cases:

(1) M is not bounded from above. Then $U_M = X_i \in T_i$.

(2) M is bounded from above. Let $m = \sup(M)$ be the least upper bound of M. Then $U_M = [\infty_i, m) \in T_i$.

Let (X, T) be the disjoint topological union of the family $\{(X_i, T_i) : i \in I\}$. Since A_i, $i \in I$, is separable, it follows that X satisfies statement **(A)**. By our hypothesis, X satisfies **(B)**, hence for each $i \in I$, let $\mathcal{C}(\infty_i)$ be a countable local base at ∞_i. For each $i \in I$, let $F_i = \{a \in A_i : [\infty_i, a) \in \mathcal{C}(\infty_i)\}$. Clearly, F_i is a countable subset of A_i for each $i \in I$. This completes the proof of the theorem. ∎

Theorem 2 *There is a model of* **ZF**^{0} *in which there exists a topological space which satisfies* **(A)** *and* $\neg$**(B)**.

Proof. The set of atoms is the set $A = \cup\{A_n : n \in \omega\}$, $A_n = \{a_{n,x} : x \in \mathbb{R}\}$. The group G of permutations of A is the set of all permutations on A which are a translation on A_n, i.e., if $\pi \in G$, then $\pi | A_n(a_{n,x}) = a_{n,y+x}$ for some $y \in \mathbb{R}$. The normal ideal I of supports is the set of all finite subsets of A. Let $(\mathcal{N}, \in)$ be the resulting permutation model which is determined by G and I.

The following hold:

(1) For each $n \in \omega$, A_n has a definable linear order in $\mathcal{N}$ which is isomorphic to the standard order of $\mathbb{R}$.
(2) The family $\mathcal{A} = \{A_n : n \in \omega\}$ does not admit in $\mathcal{N}$ a function f such that $f(n)$ is a countable subset of A_n for all $n \in \omega$.

Therefore, by Theorem 1, the implication **(A)** $\Rightarrow$ **(B)** fails in $\mathcal{N}$. ∎

Definition 2 1. The ordering principle (**OP**): *Every set can be linearly ordered.*

2. **AC($\aleph_0$)**: **AC** *restricted to families of non-empty countable sets.*

Theorem 3 *(**(B)** $\Rightarrow$ **(C)**) implies the choice principle: Every family $\mathcal{A} = \{(A_i, \leq_i) : i \in I\}$ of countable linearly ordered sets has a choice function.*
*In particular, (**OP**) + (**(B)** $\Rightarrow$ **(C)**) implies **AC($\aleph_0$)**.*

Proof. Let $\mathcal{A}$ be a family as in the statement of the theorem. Let $\{\infty_i : i \in I\}$ be a set of distinct elements which is disjoint from $\bigcup \mathcal{A}$. For each $i \in I$, let $X_i = A_i \cup \{\infty_i\}$ be the Alexandroff one-point compactification of the discrete space A_i. Let X be the disjoint topological union of the X_i's. It can be readily verified that X satisfies (B). Hence, by our hypothesis, X satisfies (C). Let for each $i \in I$, $\{\mathcal{B}(k, \infty_i) : k \in \mathbb{N}\}$ be a local base at ∞_i. Without loss of generality assume that for each $i \in I$, $\mathcal{B}(1, \infty_i) \neq X_i$. Then $f = \{(i, \leq_i - \min(X_i \backslash \mathcal{B}(1, \infty_i))) : i \in I\}$ is a choice function for the family $\mathcal{A}$. This completes the proof of the theorem. ∎

(A) $\Rightarrow$ **(B)** and **(B)** $\Rightarrow$ **(C)** are not weak axioms since the **principle of dependent choices** (**DC**), i.e., the statement:

> *Given a non-empty set X and a relation R on X such that $(\forall x \in X)(\exists y \in X)(xRy)$, there is a sequence $(x_n)_{n\in\omega}$ of elements of X such that $x_n R x_{n+1}$ for all $n \in \omega$.*

does not imply any of the above two principles in **ZF**. Moreover, none of **(A)** $\Rightarrow$ **(B)** and **(B)** $\Rightarrow$ **(C)** implies **DC** in **ZF**^{0}.

Theorem 4 **DC** *does not imply any of the following in* **ZF**^{0}*:*
(i) **(A)** $\Rightarrow$ **(B)**.
(ii) **(B)** $\Rightarrow$ **(C)**.

Proof. (i) The set of atoms is the set $A = \cup\{A_n : n \in \aleph_1\}$, $A_n = \{a_{n,x} : x \in \mathbb{R}\}$. The group G of permutations of A is the set of all permutations on A which are a translation on A_n. The normal ideal I of supports is the set of all subsets E of A with $|E| \leq \aleph_0$. Let $(\mathcal{N}, \in)$ be the permutation model which is determined by G and I. The following hold:

(1) Each A_n has a definable linear order in $\mathcal{N}$ which is isomorphic to the standard order of $\mathbb{R}$.
(2) The family $\{A_n : n \in \omega\}$ admits no function $f \in \mathcal{N}$ such that for each $n \in \omega$, $f(n)$ is a non-empty countable subset of A_n.
(3) The model $\mathcal{N}$ satisfies **DC**.

Furthermore, by **(1)**, **(2)**, and Theorem 1, we conclude that **(A)** $\Rightarrow$ **(B)** fails in the model $\mathcal{N}$. Hence, the result.

(ii) The set of atoms is the set $A = \cup\{A_n : n \in \aleph_1\}$, $A_n = \{a_{n,x} : x \in \mathbb{Q}\}$. The group G of permutations of A is the set of all permutations on A which are a **rational** translation on A_n. The normal ideal I of supports is the set of all subsets E of A with $|E| \leq \aleph_0$. Let $(\mathcal{N}, \in)$ be the permutation model which is determined by G and I. The following hold:

(1) Each A_n is countable in $\mathcal{N}$ and has a definable linear order in $\mathcal{N}$ which is isomorphic to the standard order of $\mathbb{Q}$.
(2) The family $\{A_n : n \in \omega\}$ has no choice function in $\mathcal{N}$.
(3) DC holds in $\mathcal{N}$.

Moreover, by **(1)**, **(2)**, and Theorem 3 we infer that **(B)** $\Rightarrow$ **(C)** fails in $\mathcal{N}$. Hence, the result. ∎

The interrelation between (A) $\Rightarrow$ (B) and (B) $\Rightarrow$ (C)

Definition 3 1. The Axiom of Multiple Choice (**MC**): *For every family $\mathcal{A}$ of non-empty sets there exists a function f such that for all $x \in \mathcal{A}$, $f(x)$ is a non-empty finite subset of x.*

2. **MC**$(2^{\aleph_0})$: **MC** *restricted to families of continuum sized sets.*

3. ω-**MC**: *For every family $\mathcal{A}$ of non-empty sets there exists a function f such that for all $x \in \mathcal{A}$, $f(x)$ is a non-empty countable subset of x.*

Theorem 5 *(Gutierres) (i) ω-**MC** implies (**(A)** $\Rightarrow$ **(B)**).*
*(ii) **MC**$(2^{\aleph_0})$ implies (**(B)** $\Rightarrow$ **(C)**).*

Theorem 6 *There is a model of* **ZF**^{0} *which satisfies* (**(A)** $\Rightarrow$ **(B)**) *and* $\neg$(**(B)** $\Rightarrow$ **(C)**).

Proof. For the proof of our independence result, we use the permutation model $\mathcal{N}53$ from the book:

P. Howard and J. E. Rubin, *Consequences of the Axiom of Choice*, A.M.S. Math. Surveys and Monographs, **59**, 1998.

The set of atoms is the set $A = \cup\{A_n : n \in \omega\}$, $A_n = \{a_{n,q} : q \in \mathbb{Q}\}$. The group G of permutations of A is the set of all permutations on A which are a rational translation on A_n. The normal ideal I of supports is the set of all finite subsets of A. $\mathcal{N}53$ is the resulting permutation model which is determined by G and I. The following hold:

(1) Each A_n is countable in $\mathcal{N}$ and has a definable linear order in $\mathcal{N}$ which is isomorphic to the standard order of $\mathbb{Q}$.
(2) ω-**MC** holds in the model.
(3) The family $\{A_n : n \in \omega\}$ has no choice function in $\mathcal{N}$.

By **(2)** and Theorem 5 we infer that **(A)** $\Rightarrow$ **(B)** holds true in the model.

On the other hand, by **(1)**, **(3)**, and Theorem 3, we have that **(B)** $\Rightarrow$ **(C)** fails in the model. Hence, the independence result. ∎

Corollary 1 *ω-**MC** does not imply ((**B**) $\Rightarrow$ (**C**)) in $\mathbf{ZF}^0$.*

Question. Does (**(B)** $\Rightarrow$ **(C)**) imply (**(A)** $\Rightarrow$ **(B)**) in $\mathbf{ZF}^0$?