Infinite Hausdorff spaces may lack cellular families or relatively discrete subspaces of cardinality $\aleph_0$

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Aim of this project

In set theory without the Axiom of Choice (AC), we investigate the deductive strength of "*Every infinite Hausdorff space has a countably infinite cellular family*" and "*Every infinite Hausdorff space has a countably infinite relatively discrete subspace*", and of variants of the above statements for certain classes of Hausdorff spaces, their mutual relationship, as well as their relationship with various weak choice principles.

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A couple of central results are: **(a)** the first of the above two topological principles *does not imply* the second one in ZFA (Zermelo–Fraenkel set theory with atoms); and **more strikingly** **(b)** none of the above statements is provable in ZF (Zermelo–Fraenkel set theory without AC) even for **countably infinite** Hausdorff spaces.

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$\mathrm{Iso}(X)$ denotes the set of isolated points of X, and X' $(= X \setminus \mathrm{Iso}(X))$ the set of its accumulation points.

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- IPS(cell, $\aleph_0$): Every infinite P space has a denumerable cellular family.
- IPS(reldiscr, $\aleph_0$): Every infinite P space has a denumerable relatively discrete subspace.
- DPS(cell, $\aleph_0$) and DPS(reldiscr, $\aleph_0$) stand for the above statements restricted to denumerable Hausdorff spaces.

  The property P shall be '*Hausdorff*' (H), or '*effectively Hausdorff*' (EH), or '*Compact Hausdorff*' (CH), or '*Compact Scattered Hausdorff*' (CSH) or '*Scattered Hausdorff*' (SH).

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- MC: Axiom of Multiple Choice. (MC is equivalent to AC in
- ZF, but not in ZFA.)
- DC: Principle of Dependent Choice.
- AC^{ω}: Axiom of Countable Choice.
- DF = F: Every Dedekind-finite set is finite.
- wDF = F: Every weakly Dedekind-finite set is finite. ($DC \to AC^{\omega} \to DF = F \to wDF = F$; none of the above implications is reversible in ZF.)

Results

Theorem

(ZFA) $\mathrm{MC} \rightarrow \mathrm{IHS}(\mathrm{cell}, \aleph_0) \rightarrow \mathrm{IEHS}(\mathrm{cell}, \aleph_0) \rightarrow \mathrm{wDF} = \mathrm{F}$. *Thus,* $\mathrm{IEHS}(\mathrm{cell}, \aleph_0)$ *(and hence* $\mathrm{IHS}(\mathrm{cell}, \aleph_0)$*) is not provable in* ZF.

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The following lemma, which provides an interesting characterization of MC in topological terms, shall be useful for the proof of the above theorem.

Lemma

(ZFA) *The following are equivalent:*
(i) MC*;*
(ii) For every Hausdorff space (X, τ)*, there exists a function* F *on* $Z = \{(A, B) \in ([X]^{<\omega} \setminus \{\emptyset\})^2 : A \cap B = \emptyset\}$ *such that for every* $(A, B) \in Z$, $F(A, B) = (U_A, V_B) \in \tau^2$ *with* $U_A \cap V_B = \emptyset$, $A \subset U_A$, *and* $B \subset V_B$*;*
(iii) Every Hausdorff space is effectively Hausdorff.

Proof of the lemma: (i) $\rightarrow$ (ii) Let (X, τ) be an infinite Hausdorff space. Then for all $(A, B) \in Z$,

$$S_{(A,B)} = \{(U, V) \in \tau^2 : U \cap V = \emptyset,\ A \subseteq U,\ B \subseteq V\} \neq \emptyset.$$

Let G be a multiple choice function for the family

$$\mathcal{A} = \{S_{(A,B)} : (A, B) \in Z\}.$$

For each $(A, B) \in Z$, set
$U_A = \bigcap\{\pi_1((O, Q)) : (O, Q) \in G(S_{(A,B)})\}$ and
$V_B = \bigcap\{\pi_2((O, Q)) : (O, Q) \in G(S_{(A,B)})\}$, where π_1 and π_2 are the canonical projections on the first and second coordinates, respectively. Then U_A and V_B are open and disjoint with $A \subseteq U_A$ and $B \subseteq V_B$.
Letting for each $(A, B) \in Z$, $F(A, B) = (U_A, V_B)$, we conclude that F is the required function.

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(ii) $\rightarrow$ (iii) This is clear.

Results

(iii) $\rightarrow$ (i) Let $\mathcal{A} = \{A_i : i \in I\}$ be a family of non-empty sets. Wlog assume that $\mathcal{A}$ is disjoint and that for all $i \in I$, A_i is infinite. Let $A = \{a_i : i \in I\}$ and $B = \{b_i : i \in I\}$ be two sets with $A \cap B \cap \bigcup \mathcal{A} = \emptyset$. For each $i \in I$, let

$$X_i = [A_i]^{<\omega} \cup \{a_i, b_i\}$$

and let τ_i be the topology on X_i which is generated by the family β_i comprising the following sets:

1. $\{x\}$, $x \in [A_i]^{<\omega}$,
2. $M(a_i, y) = \{a_i\} \cup \{x \in [A_i]^{<\omega} : y \subseteq x\}$, $y \in [A_i]^{<\omega}$,
3. $N(b_i, z) = \{b_i\} \cup \{x \in [A_i]^{<\omega} : x \cap z = \emptyset\}$, $z \in [A_i]^{<\omega}$.

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Then for all $i \in I$, (X_i, τ_i) is a Hausdorff space. Thus, $X = \bigcup\{X_i : i \in I\}$ with the disjoint union topology is (by our hypothesis) effectively Hausdorff. Let F be a function such that for all $i \in I$, $F(a_i, b_i) = (U_{a_i}, V_{b_i})$, where U_{a_i} and V_{b_i} are disjoint open neighborhoods of a_i and b_i, respectively.

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For each $i \in I$, let

$$P_{a_i} = U_{a_i} \cap X_i, \; P_{b_i} = V_{b_i} \cap X_i,$$

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For each $i \in I$, either $\mathcal{U}_{i,n_i}$ is finite, or there exists a (least) natural number k_i such that the family $\mathcal{R}_{i,k_i} = \{r \in [\bigcup \mathcal{V}_i]^{k_i} : \forall v \in \mathcal{V}_i, r \cap v \neq \emptyset\}$ is a non-empty finite set.

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Then the mapping

$$G(i) = \begin{cases} \bigcup \mathcal{U}_{i,n_i}, & \text{if } \mathcal{U}_{i,n_i} \text{ is finite} \\ \bigcup \mathcal{R}_{i,k_i}, & \text{if } \mathcal{U}_{i,n_i} \text{ is infinite} \end{cases}$$

is a multiple choice function of $\mathcal{A}$. Thus, MC holds as required. □

Remark

A stronger result than the previous lemma has been obtained by Howard–Keremedis–Rubin–Rubin [Math. Log. Quart. bf 44 (1998), 367–382], namely MC is equivalent to "Every T_4 space is effectively T_4".

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Proof of the theorem: Assume MC. Let (X, τ) be an infinite Hausdorff topological space. There are two cases:

Case 1. $\mathrm{Iso}(X)$ *is infinite*. By the previous lemma, $\mathrm{Iso}(X) = \bigcup\{I_\alpha : \alpha < \kappa\}$, where κ is an infinite well-ordered cardinal number and $\{I_\alpha : \alpha < \kappa\}$ is a disjoint family of (non-empty) finite sets. Clearly, $\{I_\alpha : \alpha < \aleph_0\}$ is a denumerable cellular family in X.

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Case 2. $\mathrm{Iso}(X)$ *is finite*. Then X' is open in X, so it suffices to find a cellular family in the dense-in-itself subspace X' of X. By the lemma, X' has a well-ordered partition $\{Y_\alpha : \alpha < \kappa\}$ into finite sets.

Results

By the first lemma, let F be a Hausdorff operator.

Let $F(Y_0, Y_1) = (U_{Y_0}, V_{Y_1})$ and set $U_0 = U_{Y_0}$. Since X' is dense-in-itself, V_{Y_1} is infinite, hence let m_1 be the least ordinal $m \in \kappa \setminus \{0, 1\}$ such that $V_{Y_1} \cap Y_m \neq \emptyset$.

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We continue in this fashion by induction, thus obtaining a denumerable cellular family $\mathcal{U} = \{U_n : n \in \omega\}$ in X', and hence in X.

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Corollary

$\mathrm{IHS}(\mathrm{cell}, \aleph_0)$ *does not imply* $\mathrm{AC}^{\omega}_{\mathrm{fin}}$ *(*AC *for denumerable families of non-empty finite sets) in* ZFA*, and hence it does not imply* $\mathrm{DF} = \mathrm{F}$ *in* ZFA *either.*

Results

Theorem

$\mathsf{AC}^\omega \rightarrow \mathsf{IHS}(\mathrm{cell}, \aleph_0) + \mathsf{IHS}(\mathrm{reldiscr}, \aleph_0)$. *Hence* $\mathsf{IHS}(\mathrm{cell}, \aleph_0)$ *is strictly weaker than each of* AC^ω *and* MC *in* ZFA.

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Proof. Assume that AC^ω is true. Let (X, τ) be an infinite Hausdorff space. If $\mathrm{Iso}(X)$ is infinite, then any denumerable subset of $\mathrm{Iso}(X)$ (guaranteed by AC^ω) yields a denumerable cellular family. So wlog we assume that X is dense-in-itself (otherwise we work with X'). For each $n \in \omega$, let

$$A_n = \{(U_0, U_1, \ldots, U_n) : \{U_0, U_1, \ldots, U_n\} \text{ is a cellular family in } X\}.$$

Since X is Hausdorff and dense-in-itself, X has arbitrarily large finite cellular families, and thus, $A_n \neq \emptyset$ for all $n \in \omega$. By AC^ω, let

$$f = \{(n, (U_0^n, U_1^n, \ldots, U_n^n)) : n \in \omega\}$$

be a choice function of the family $\mathcal{A} = \{A_n : n \in \omega\}$.

Results

Via mathematical induction, we construct a denumerable cellular family in X. Put $W_0 = U_0^0$. Assume that we have chosen pairwise disjoint non-empty open sets $W_0, W_1, \ldots, W_n$, where n is some natural number. Consider the $(n+1)$-tuple

$$f(n+1) = (U_0^{(n+1)}, U_1^{(n+1)}, \ldots, U_{n+1}^{(n+1)})$$

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Case 1. $\exists j \leq n+1,\ U_j^{(n+1)} \cap \bigcup_{i \leq n} W_i = \emptyset$. Let j_{n+1} be the least such $j \leq n+1$ and let $W_{n+1} = U_{j_{n+1}}^{(n+1)}$. Then $W_{n+1} \cap W_i = \emptyset$ for all $i \leq n$.

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Case 2. $\forall j \leq n+1,\ U_j^{(n+1)} \cap \bigcup_{i \leq n} W_i \neq \emptyset$. Since $|\operatorname{ran}(f(n+1))| = n+2 > n+1 = |\{W_0, W_1, \ldots, W_n\}|$, it follows that for some $i \leq n$, we have that $|S_i| \geq 2$, where

$$S_i = \{j \leq n+1 : U_j^{(n+1)} \cap W_i \neq \emptyset\}.$$

Let i_{n+1} be the least such $i \leq n$ and let $j_{n+1} = \min(S_{i_{n+1}})$ and $k_{n+1} = \min(S_{i_{n+1}} \setminus \{j_{n+1}\})$. Now, let

$$W_{i_{n+1}} = U^{(n+1)}_{j_{n+1}} \cap W_{i_{n+1}}$$

(that is, we replace the old $W_{i_{n+1}}$ in the sequence $(W_0, W_1, \ldots, W_n)$ by $U_{j_{n+1}} \cap W_{i_{n+1}}$ and we label the latter (non-empty) intersection as $W_{i_{n+1}}$ again) and also let

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Clearly, $\{W_0, W_1, \ldots, W_{i_{n+1}}, \ldots, W_n, W_{n+1}\}$ is a cellular family in X. This completes the inductive step and the proof of the implication.

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Clearly, $\{W_0, W_1, \ldots, W_{i_{n+1}}, \ldots, W_n, W_{n+1}\}$ is a cellular family in X. This completes the inductive step and the proof of the implication.

For the second implication, if $\mathcal{O}$ is a denumerable cellular family in X, then (by AC^{ω}) any choice set of $\mathcal{O}$ is a denumerable relatively discrete subset of X. $\square$

Results

Problem

Does $\mathrm{DF} = \mathrm{F}$ *imply any of* $\mathrm{IHS}(\mathrm{cell}, \aleph_0)$, $\mathrm{IEHS}(\mathrm{cell}, \aleph_0)$, *and* $\mathrm{IHS}(\mathrm{reldiscr}, \aleph_0)$*?*

We give partial answers to the above questions.

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Theorem

ISHS(reldiscr, $\aleph_0$) $\iff$ ICSHS(reldiscr, $\aleph_0$) $\iff$ DF = F.

Results

Problem

Does $\mathrm{DF} = \mathrm{F}$ *imply any of* $\mathrm{IHS}(\mathrm{cell}, \aleph_0)$, $\mathrm{IEHS}(\mathrm{cell}, \aleph_0)$, *and* $\mathrm{IHS}(\mathrm{reldiscr}, \aleph_0)$*?*

We give partial answers to the above questions.

Theorem

$\mathrm{ISHS}(\mathrm{reldiscr}, \aleph_0) \iff \mathrm{ICSHS}(\mathrm{reldiscr}, \aleph_0) \iff \mathrm{DF} = \mathrm{F}$.

Corollary

$\mathrm{IHS}(\mathrm{cell}, \aleph_0) \nrightarrow \mathrm{IHS}(\mathrm{reldiscr}, \aleph_0)$ *in* ZFA.

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$\mathrm{IHS}(\mathrm{cell}, \aleph_0) \nrightarrow \mathrm{IHS}(\mathrm{reldiscr}, \aleph_0)$ *in* ZFA.

Theorem

$\mathrm{DF} = \mathrm{F} \rightarrow \mathrm{ISHS}(\mathrm{cell}, \aleph_0)$. *The implication is not reversible in* ZFA.

Results

Theorem

"Every infinite zero-dimensional Hausdorff space has a denumerable relatively discrete subspace" $\rightarrow \mathsf{DF} = \mathsf{F} \rightarrow$ *"Every infinite zero-dimensional Hausdorff space has a denumerable cellular family"* $\rightarrow \mathsf{ICSHS}(\mathrm{cell}, \aleph_0) \rightarrow \mathsf{wDF} = \mathsf{F}$.

Results

Theorem

"Every infinite zero-dimensional Hausdorff space has a denumerable relatively discrete subspace" $\rightarrow$ DF = F $\rightarrow$ *"Every infinite zero-dimensional Hausdorff space has a denumerable cellular family"* $\rightarrow$ ICSHS$(\text{cell}, \aleph_0)$ $\rightarrow$ wDF = F.

The last but one implication of the above theorem follows from the **ZF-result** of Keremedis–Felouzis–T. [Bull. Polish Acad. Sci. Math. **54** (2006), 75–84] that *every compact scattered Hausdorff space is zero-dimensional*. This was originally proved by Ostaszewski [J. London Math. Soc. **14** (1976), 505–516]; however, the proof employed the axiom of choice.

Results

Theorem

If $\mathcal{N}$ *is a Fraenkel–Mostowski permutation model of* ZFA *such that*

$$\mathcal{N} \models \mathsf{DF} = \mathsf{F},$$

then

$$\mathcal{N} \models \mathsf{IHS}(\mathsf{reldiscr}, \aleph_0).$$

Hence $\mathsf{IHS}(\mathsf{reldiscr}, \aleph_0)$ *is strictly weaker than* AC^ω *in* ZFA*.*

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Hence $\mathsf{IHS}(\mathsf{reldiscr}, \aleph_0)$ *is strictly weaker than* AC^ω *in* ZFA.

Theorem

The following hold:
(1) "Every infinite dense-in-itself Hausdorff space has a denumerable relatively discrete subspace" $\rightarrow \mathsf{DF} = \mathsf{F}|_{\mathbb{R}} + \mathsf{wDF} = \mathsf{F}$.
(2) "Every infinite dense-in-itself Hausdorff space has a denumerable relatively discrete subspace" $+ \mathsf{AC}^\omega_{\mathsf{fin}} \rightarrow \mathsf{DF} = \mathsf{F}$.

Proof of (1). Let A be an infinite set of reals. Set $X = [A]^{<\omega}$, and let $\mathcal{B}$ be the family of all sets of the form

$$N(x, w) = \{y \in X : x \subseteq y \text{ and } y \cap w = \emptyset\},$$

where $x, w \in X$ with $x \cap w = \emptyset$. Then $\mathcal{B}$ is a base for a dense-in-itself Hausdorff topology $\tau_{\mathcal{B}}$ on X:

1. $X = \bigcup \mathcal{B}$, and $N(x_1, w_1) \cap N(x_2, w_2) = N(x_1 \cup x_2, w_1 \cup w_2)$;
2. Every non-empty member of $\mathcal{B}$ is infinite;
3. Let $x, y \in X$ with $x \neq y$. (a) $x \cap y = \emptyset$; then $N(x, y) \cap N(y, x) = \emptyset$; (b) $x \subsetneq y$; then $N(x, y \setminus x) \cap N(y, z) = \emptyset$, for any $z \in X$ with $z \cap y = \emptyset$; (c) $y \subsetneq x$; similar to (b); (d) $x \not\subseteq y$ and $y \not\subseteq x$; then $N(x, y \setminus x) \cap N(y, x \setminus y) = \emptyset$. Thus, X is Hausdorff.

Thus, X has (by hypothesis) a denumerable relatively discrete subspace, say Y. Since $Y \subset [\mathbb{R}]^{<\omega}$, it follows that $\bigcup Y \in [A]^{\omega}$. $\square$

Results on countably infinite Hausdorff spaces

Theorem

The following hold:

(i) $\mathsf{DSHS}(\mathrm{cell}, \aleph_0)$ *and* $\mathsf{DSHS}(\mathrm{reldiscr}, \aleph_0)$ *are provable in* ZF*.*

(ii) $\mathsf{AC}^{\omega}_{\mathbb{R}} \rightarrow \mathsf{DHS}(\mathrm{cell}, \aleph_0) \rightarrow \mathsf{DHS}(\mathrm{reldiscr}, \aleph_0)$. *Thus,* $\mathsf{DHS}(\mathrm{cell}, \aleph_0)$ *(and hence* $\mathsf{DHS}(\mathrm{reldiscr}, \aleph_0)$*) is true in every Fraenkel–Mostowski permutation model of* ZFA*.*

(iii) $\mathsf{AC}^{\omega}_{\mathbb{R}} \rightarrow$ *"every denumerable compact Hausdorff space is metrizable"* $\rightarrow \mathsf{DCHS}(\mathrm{cell}, \aleph_0) \rightarrow \mathsf{DCHS}(\mathrm{reldiscr}, \aleph_0)$.

Results on countably infinite Hausdorff spaces

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The following hold:

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(ii) $\mathsf{AC}_{\mathbb{R}}^{\omega} \rightarrow \mathsf{DHS}(\mathrm{cell}, \aleph_0) \rightarrow \mathsf{DHS}(\mathrm{reldiscr}, \aleph_0)$. *Thus,* $\mathsf{DHS}(\mathrm{cell}, \aleph_0)$ *(and hence* $\mathsf{DHS}(\mathrm{reldiscr}, \aleph_0)$*) is true in every Fraenkel–Mostowski permutation model of* ZFA.

(iii) $\mathsf{AC}_{\mathbb{R}}^{\omega} \rightarrow$ *"every denumerable compact Hausdorff space is metrizable"* $\rightarrow \mathsf{DCHS}(\mathrm{cell}, \aleph_0) \rightarrow \mathsf{DCHS}(\mathrm{reldiscr}, \aleph_0)$.

The implication '$\mathsf{AC}_{\mathbb{R}}^{\omega} \rightarrow$ "every denumerable compact Hausdorff space is metrizable"' has been established in Keremedis–T. [Proc. Amer. Math. Soc. **135** (2007), 1205–1211], where it is also shown that it is relatively consistent with ZF that there exists a denumerable compact zero-dimensional Hausdorff space (with countably infinite cellular families) which is not metrizable.

Results on countably infinite Hausdorff spaces

Theorem

There is a model N of ZF *such that*

$$N \models \neg \mathrm{DHS}(\mathrm{reldiscr}, \aleph_0).$$

Hence, $\mathrm{DHS}(\mathrm{reldiscr}, \aleph_0)$ *(and consequently* $\mathrm{DHS}(\mathrm{cell}, \aleph_0)$*) is not provable in* ZF *set theory.*

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Proof. We start with a countable transitive model M of ZFC. By forcing with finite partial functions from $\omega \times \omega$ into 2, i.e., with $\mathbb{P} = \mathrm{Fn}(\omega \times \omega, 2)$ partially ordered by reverse inclusion, we obtain a generic model $M[G] \supset M$, where G is a $\mathbb{P}$-generic set over M, with a countably infinite set $A = \{a_n : n \in \omega\}$ of generic reals, i.e. of generic subsets of ω. (For $n \in \omega$,
$a_n = \{m \in \omega : \exists p \in G, p(n, m) = 1\}$, and
$a_n^c = \omega \setminus a_n = \{m \in \omega : \exists p \in G, p(n, m) = 0\}$.)

Results on countably infinite Hausdorff spaces

In $M[G]$, we let $\mathcal{B}$ be the Boolean subalgebra of $(\mathcal{P}(\omega), \triangle, \cap)$ which is generated by the set A (i.e., a set $x \subset \omega$ in $M[G]$ is in $\mathcal{B}$ if and only if x is written as a finite union of finite intersections of elements and complements of elements from A).

Let

$$N = \mathrm{HOD}(\mathcal{B}),$$

i.e., N is the submodel of $M[G]$ which consists of all elements of $M[G]$ which are hereditarily ordinal-definable over $\mathcal{B}$ in $M[G]$, that is, an element $x \in M[G]$ belongs to N if and only if for every $z \in \{x\} \cup \mathrm{TC}(x)$ (where $\mathrm{TC}(x)$ is the transitive closure of x) there is a formula φ in the language of set theory such that, in $M[G]$,

$$z = \{u : \varphi(u, \alpha_1, \ldots, \alpha_n, b_1, \ldots, b_k, \mathcal{B})\}$$

for some ordinal numbers $\alpha_1, \ldots, \alpha_n$ and for finitely many elements $b_1, \ldots, b_k$ in $\mathcal{B}$. N is a transitive model of ZF. Furthermore, $\mathcal{B} \in N$, $A \notin N$, and $\mathcal{B}$ is not well-orderable in N.

Results on countably infinite Hausdorff spaces

It is clear that $\mathcal{B}$ is a base for a topology $\tau_{\mathcal{B}}$ on ω.

Lemma

$(\omega, \tau_{\mathcal{B}})$ *is a dense-in-itself zero-dimensional Hausdorff space in* N.

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$(\omega, \tau_{\mathcal{B}})$ is a dense-in-itself zero-dimensional Hausdorff space in N.

Proof of lemma. A is an independent family in $M[G]$, hence every non-empty member of $\mathcal{B}$ is infinite, and thus ω is dense-in-itself. Moreover, since $\mathcal{B}$ is a Boolean algebra, $(\omega, \tau_{\mathcal{B}})$ is 0-dimensional. Let $n, m \in \omega$ with $n \neq m$, and also let

$$D_{n,m} = \{p \in \mathbb{P} : (\exists k \in \omega)((k, n) \in \mathrm{dom}(p),$$

$$(k, m) \in \mathrm{dom}(p), \text{ and } p(k, n) = 1, p(k, m) = 0)\}.$$

Then $D_{n,m}$ is dense in $\mathbb{P}$ and belongs to M, thus $G \cap D_{n,m} \neq \emptyset$. Thus, $\exists k \in \omega$ such that $n \in a_k$ and $m \in a_k^c$. Since a_k and a_k^c belong to $\mathcal{B}$, it follows that a_k and a_k^c are disjoint open neighborhoods of n and m. $\square$

Lemma

$(\omega, \tau_{\mathcal{B}})$ *has no infinite relatively discrete subset in* N.

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$(\omega, \tau_{\mathcal{B}})$ has no infinite relatively discrete subset in N.

Proof of lemma. Assume the contrary and let $Y \subset \omega$ be an infinite relatively discrete subset of ω in N. Since N is a model of ZF, it follows that

$$W = \{(n, b) : n \in Y, b \in \tau_{\mathcal{B}} \text{ and } b \cap Y = \{n\}\} \in N.$$

Thus, there exist $\beta_1, \ldots, \beta_m \in \mathrm{Ord}$ and elements $c_1, \ldots, c_k \in \mathcal{B}$ such that W is definable in $M[G]$ by a formula ψ and the parameters $\beta_1, \ldots, \beta_m$, $c_1, \ldots, c_k, \mathcal{B}$, i.e.,

$$(n, b) \in W \iff M[G] \models \psi(\beta_1, \ldots, \beta_m, c_1, \ldots, c_k, \mathcal{B}, n, b). \quad (1)$$

Let $\mathcal{B}^*$ be the Boolean subalgebra of $\mathcal{B}$ which is generated by the finite set Z of Cohen reals in the expressions of the c_r, $r = 1, \ldots, k$. Since $\mathcal{B}^*$ is finite, and Y is infinite and relatively discrete, there is an $n \in Y$, a Cohen real a_{i^*} such that neither a_{i^*} nor $a^c_{i^*}$ appears in Z, and an open set U which is a finite intersection of elements and complements of elements from A, and such that a_{i^*} appears in the expression of U, and $U \cap Y = \{n\}$. Thus

$$M[G] \models \psi(\beta_1, \ldots, \beta_m, c_1, \ldots, c_k, \mathcal{B}, n, U) \wedge (U \subseteq a_{i^*}). \quad (2)$$

It follows that there is a forcing condition $p \in G$ such that

$$p \Vdash \psi(\check{\beta}_1, \ldots, \check{\beta}_m, \dot{c}_1, \ldots, \dot{c}_k, \dot{\mathcal{B}}, \check{n}, \dot{U}) \wedge (\dot{U} \subseteq \dot{a}_{i^*}) \quad (3)$$

(where $\dot{c}_r$, $r = 1, \ldots, k$, is the canonical name of c_r).

Results on countably infinite Hausdorff spaces

Since p is finite, let $m_0 \in \omega$ such that

$$\forall m \in \omega (m \geq m_0 \to (i^*, m) \notin \operatorname{dom}(p)), \tag{4}$$

and let

$$X = \{(i^*, m) : m \in \omega, m \geq m_0\}.$$

Define $\pi_X : \mathbb{P} \to \mathbb{P}$ by

$$\pi_X(s)(u,v) = \begin{cases} s(u,v), & \text{if } (u,v) \notin X \\ 1 - s(u,v), & \text{if } (u,v) \in X, \end{cases}$$

π_X is an order automorphism of the poset $(\mathbb{P}, \supseteq)$, and induces an $\in$-automorphism of $M^{\mathbb{P}}$ (the set of $\mathbb{P}$-names). From (4) and the definitions of X and π_X, we conclude that π_X has the following properties:

Results on countably infinite Hausdorff spaces

1. $\pi_X(p) = p$,
2. π_X fixes the canonical names $\dot{c}_r$, $r = 1, \ldots, k$, and $\pi_X(\dot{\mathcal{B}}) = \dot{\mathcal{B}}$,
3. $a_{i^*} \cap (\pi_X(\dot{a_{i^*}}))_G \subseteq m_0$ (where $(\pi_X(\dot{a_{i^*}}))_G$ is the value of the name $\pi_X(\dot{a_{i^*}})$), thus $a_{i^*} \cap (\pi_X(\dot{a_{i^*}}))_G$ is a finite set.

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Applying π_X to (3) gives

$$p \Vdash \psi(\check{\beta}_1, \ldots, \check{\beta}_m, \dot{c}_1, \ldots, \dot{c}_k, \dot{\mathcal{B}}, \check{n}, \pi_X(\dot{U})) \wedge (\pi_X(\dot{U}) \subseteq \pi_X(\dot{a_{i^*}})). \tag{5}$$

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Since $p \in G$, we conclude that

$$M[G] \models \psi(\beta_1, \ldots, \beta_m, c_1, \ldots, c_k, \mathcal{B}, n, (\pi_X(\dot{U}))_G) \wedge ((\pi_X(\dot{U}))_G \subseteq (\pi_X(\dot{a_{i^*}}))_G). \tag{6}$$

From equations (1) and (6), $(n, (\pi_X(\dot{U}))_G) \in W$ and hence

$$(\pi_X(\dot{U}))_G \in \tau_{\mathcal{B}} \text{ and } (\pi_X(\dot{U}))_G \cap Y = \{n\}.$$

Therefore,

$$U \cap (\pi_X(\dot{U}))_G \neq \emptyset.$$

Since $U \cap (\pi_X(\dot{U}))_G$ is open and ω is dense-in-itself,

$$U \cap (\pi_X(\dot{U}))_G \text{ is infinite.} \tag{7}$$

However, since $U \subseteq a_{i^*}$ and $(\pi_X(\dot{U}))_G \subseteq (\pi_X(\dot{a_{i^*}}))_G$, we infer that

$$U \cap (\pi_X(\dot{U}))_G \subseteq a_{i^*} \cap (\pi_X(\dot{a_{i^*}}))_G \subseteq m_0,$$

hence $U \cap (\pi_X(\dot{U}))_G$ is finite, contradicting (7). Thus, ω has no infinite relatively discrete subspaces in the model N. $\square$

Open problems

1. Does $\mathrm{DF} = \mathrm{F}$ imply any of $\mathrm{IHS}(\mathrm{cell}, \aleph_0)$ and $\mathrm{IEHS}(\mathrm{cell}, \aleph_0)$?
2. Does $\mathrm{IHS}(\mathrm{cell}, \aleph_0)$ imply $\mathrm{IdisHS}(\mathrm{reldiscr}, \aleph_0)$? Does it imply $\mathrm{DF} = \mathrm{F}|_{\mathbb{R}}$?
3. Does $\mathrm{DF} = \mathrm{F}$ imply any of $\mathrm{IHS}(\mathrm{reldiscr}, \aleph_0)$ and $\mathrm{I0dimHS}(\mathrm{reldiscr}, \aleph_0)$?
4. Does $\mathrm{IdisHS}(\mathrm{reldiscr}, \aleph_0)$ imply $\mathrm{DF} = \mathrm{F}$?
5. Does MC imply $\mathrm{IdisHS}(\mathrm{reldiscr}, \aleph_0)$?
6. Does $\mathrm{IHS}(\mathrm{reldiscr}, \aleph_0)$ imply $\mathrm{IHS}(\mathrm{cell}, \aleph_0)$?
7. Is $\mathrm{DCHS}(\mathrm{reldiscr}, \aleph_0)$ false in the ZF-model N?
8. Does $\mathrm{DCHS}(\mathrm{reldiscr}, \aleph_0)$ imply $\mathrm{DCHS}(\mathrm{cell}, \aleph_0)$?

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Thank You!