Martin's Axiom and Choice Principles

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Statement of Martin's Axiom

Let κ be an infinite well-ordered cardinal number. MA(κ) stands for the principle:

If $(P, \leq)$ is a non-empty c.c.c. partial order and if $\mathcal{D}$ is a family of $\leq \kappa$ dense sets in P, then there is a filter F of P such that $F \cap D \neq \emptyset$ for all $D \in \mathcal{D}$.

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Martin's Axiom: $\forall\ \omega \leq \kappa < 2^{\aleph_0}$ (MA(κ)).

Some Known Facts

- ZFC $\nvdash$ MA: **(a)** AC (the *Axiom of Choice*) + MA $\Rightarrow 2^{\aleph_0}$ is regular **(b)** it is relatively consistent with ZFC that $2^{\aleph_0}$ is singular.

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- MA$(2^{\aleph_0})$ is false.
- (ZF) DC $\Rightarrow$ MA$(\aleph_0)$ $\Rightarrow$ "every compact c.c.c. T_2 space is Baire" $\Rightarrow$ "every countable compact T_2 space is Baire", where DC is the *Principle of Dependent Choice*: if R is a binary relation on a non-empty set E such that $\forall x \in E\ \exists y \in E(x\ R\ y)$, then there is a sequence $(x_n)_{n\in\omega}$ of elements of E such that $\forall n \in \omega(x_n\ R\ x_{n+1})$.

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- MA($\aleph_0$) is **not provable** in ZF.
- (ZFC) For any $\kappa \geq \omega$, MA(κ) $\Leftrightarrow$ MA(κ) restricted to complete Boolean algebras $\Leftrightarrow$ MA(κ) restricted to partial orders of cardinality $\leq \kappa \Leftrightarrow$ if X is any compact c.c.c. T_2 space and U_α are dense open sets for $\alpha < \kappa$, then $\bigcap_\alpha U_\alpha \neq \emptyset$.

Let MA_κ denote $\mathsf{MA}(\kappa)$ restricted to partial orders of cardinality $\leq \kappa$ and let MA^* denote $\forall \kappa < 2^{\aleph_0}(\mathsf{MA}_\kappa)$. Then from the above observations we have that

$$\mathsf{ZFC} \vdash \mathsf{MA} \Leftrightarrow \mathsf{MA}^*.$$

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$\mathsf{MA}^* + \neg\mathsf{MA}(\aleph_0)$ *is relatively consistent with* ZFA.

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- Note that $\mathsf{MA}_{\aleph_0}$ is provable in ZF, $\mathsf{MA}_{\aleph_1}$ is **not** provable in ZFC (Gödel's model $L \models \mathsf{GCH} + \neg\mathsf{MA}_{2^{\aleph_0}}$), and $\mathsf{CH} \Rightarrow \mathsf{MA}^*$.

A few problems – some settled in this project

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6. Does $\forall \mathfrak{p}(2\mathfrak{p} = \mathfrak{p})$ imply MA($\aleph_0$)?

- It is **unknown** whether "*every countable compact T_2 space is Baire*" is provable in ZF. Our conjecture is that the answer is in the negative.
  We note that the stronger statement "*every countable compact T_2 space is metrizable*" is **not provable** in ZF (Keremedis–Tachtsis, 2007)

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- (Fossy–Morillon, 1998) "*Every compact T_2 space is Baire*" is **equivalent** to *Dependent Multiple Choice* (DMC): if R is a binary relation on a non-empty set E such that $\forall x \in E\ \exists y \in E(x\ R\ y)$, then there is a sequence $(F_n)_{n\in\omega}$ of non-empty finite subsets of E such that $\forall n \in \omega\ \forall x \in F_n\ \exists y \in F_{n+1}(x\ R\ y)$.

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  - DMC is strictly weaker than DC and MC (the *Axiom of Multiple Choice*).
  - MC is equivalent to AC in ZF, but is *not* equivalent to AC in ZFA.

A preliminary and a couple of known results

Theorem

"Every compact c.c.c. T_2 space is Baire" + the Boolean Prime Ideal Theorem (BPI) $\Rightarrow$ MA($\aleph_0$) *restricted to complete Boolean algebras. Thus,* DMC + BPI $\Rightarrow$ MA($\aleph_0$) *restricted to complete Boolean algebras.*

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(BPI is **equivalent** to "$\forall$ infinite X, the Stone space $S(X)$ of X is compact" (Herrlich–Keremedis–Tachtsis, 2011). We establish that BPI **cannot be dropped** from the hypotheses. Hence, MA($\aleph_0$) is **not equivalent** to "every compact c.c.c. T_2 space is Baire" in set theory without choice.)

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Proof Let $(B, +, \cdot, 0, 1)$ be a c.c.c. complete Boolean algebra. Let $S(B)$ be the Stone space of B (which is T_2). By BPI, $S(B)$ is compact. Using the fact that B has the c.c.c. and is *complete*, one shows that $S(B)$ is a c.c.c. space, and hence it is Baire. Then, a generic filter for a given countable set of dense subsets of $B \setminus \{0\}$ can be obtained, using the fact that $S(B)$ is Baire. □

Lemma

Let $(P, \leq)$ *be a partial order. Then there is a complete Boolean algebra* B *and a map* $i : P \to B \setminus \{0\}$ *such that:*

1. $i[P]$ *is dense in* $B \setminus \{0\}$*.*
2. $\forall p, q \in P \ (p \leq q \to i(p) \leq i(q))$*.*
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(If $(P, \leq)$ is a partial order, then B is the complete Boolean algebra $\mathrm{ro}(P)$ of the regular open subsets of P (O is regular open if $O = \mathrm{int}\,\mathrm{cl}(O) =$ the interior of the closure of O), where P is endowed with the topology generated by the sets $N_p = \{q \in P : q \leq p\}$, $p \in P$. Also, for $b, c \in B$, $b \leq c$ if and only if $b \subseteq c$, $b \wedge c = b \cap c$, $b \vee c = \mathrm{int}\,\mathrm{cl}(b \cup c)$, $b' = \mathrm{int}(P \setminus b)$, and if $S \subseteq B$, $\bigvee S = \mathrm{int}\,\mathrm{cl}(\bigcup S)$ and $\bigwedge S = \mathrm{int}(\bigcap S)$. For $p \in P$, $i(p) := \mathrm{int}\,\mathrm{cl}(N_p)$.)

Theorem

(Herrlich–Keremedis, 1999) The following hold:

1. $\mathrm{MA}(\aleph_0) + \mathrm{AC}^{\aleph_0}_{\mathrm{fin}}$ *implies* $\forall$ *infinite* X(2^X *is Baire) which in turn implies the following:*

   (a) $\forall$ *infinite* X, $\mathcal{P}(X)$ *is Dedekind-infinite,*

   (b) $\mathrm{AC}^{\aleph_0}_{\mathrm{fin}}$,

   (c) The Partial Kinna–Wagner Selection Principle (i.e. for every infinite family $\mathcal{A}$ *such that* $\forall X \in \mathcal{A}$, $|X| \geq 2$, *there is an infinite subfamily* $\mathcal{B}$ *and a function* F *on* $\mathcal{B}$ *such that* $\forall B \in \mathcal{B}$, $\emptyset \neq f(B) \subsetneq B$*).*

2. *For any infinite set* X, *if* 2^X *is Baire then* X *is not amorphous.*

(An infinite set X is *amorphous* if it cannot be written as a disjoint union of two infinite sets.)

Main Results

Lemma

Let A and B be two sets such that B has at least two elements. Then for $(P, \leq) = (\mathrm{Fn}(A, B), \supseteq)$, the mapping $i : P \to \mathrm{ro}(P) \setminus \{\emptyset\}$ ($i(p) = \mathrm{int}\,\mathrm{cl}(N_p)$) is $i(p) = N_p$ for all $p \in P$, where for $p \in P$, $N_p = \{q \in P : q \leq p\}$.

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Proof Fix $p \in P$. Since $\forall q \in P$, N_q is the smallest (w.r.t. $\subseteq$) open set containing q, we have $q \in \mathrm{cl}(N_p)$ iff $N_q \cap N_p \neq \emptyset$ iff p and q are compatible. Thus, $\mathrm{cl}(N_p) = \{q \in P : q$ is compatible with $p\}$. Hence, $r \in \mathrm{int}\,\mathrm{cl}(N_p)$ iff $N_r \subseteq \mathrm{cl}(N_p)$ iff every $q \leq r$ is compatible with p. Thus, $\mathrm{int}\,\mathrm{cl}(N_p) = \{r \in P :$ every extension of r is compatible with $p\}$.

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Now, let $r \in \mathrm{int}\,\mathrm{cl}(N_p)$. If $r \notin N_p$, then $p \not\subseteq r$. Then $\exists a \in A$ such that $(a, p(a)) \in p \setminus r$, and since $|B| \geq 2$, $\exists b \in B \setminus \{p(a)\}$. Let $r' = r \cup \{(a, b)\}$. Then r' is an extension of r which is incompatible with p, and hence $r \notin \mathrm{int}\,\mathrm{cl}(N_p)$, a contradiction. Therefore, $\mathrm{int}\,\mathrm{cl}(N_p) = N_p$, so $i(p) = N_p$ for all $p \in P$. □

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Proof Let $D = \{d_n : n \in \omega\}$ be a countable dense subset of $2^{\mathbb{R}}$. ($2^{\mathbb{R}}$ is separable in ZF). Let $\mathcal{B} = \mathrm{ro}(P)$ be the complete Boolean algebra associated with the poset $(P, \leq) = (\mathrm{Fn}(\mathbb{R}, 2), \supseteq)$ via the mapping i of the lemma.

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- $\mathcal{B}$ has the c.c.c.: Let S be an antichain in $\mathcal{B}$ and $s \in S$. Since $i[P]$ is dense in $\mathcal{B} \setminus \{\emptyset\}$ and D is dense in $2^{\mathbb{R}}$, we may let $n_s = \min\{n \in \omega : \exists F_{n,s} \in [\mathbb{R}]^{<\omega}, i(d_n \restriction F_{n,s}) \subseteq s\}$. Since $p \leq q \to i(p) \leq i(q)$, the map $s \mapsto n_s$, $s \in S$, is 1-1 (if $s, s' \in S$ are such that $s \neq s'$, but $n_s = n_{s'} = k$ for some $k \in \omega$, then there are $F_{k,s}, F_{k,s'} \in [\mathbb{R}]^{<\omega}$ such that

$$i(d_k \restriction F_{k,s}) \subseteq s \text{ and } i(d_k \restriction F_{k,s'}) \subseteq s'.$$

Letting q be the union of the above two restrictions of d_k we have that $i(q) \subseteq s$ and $i(q) \subseteq s'$, and thus s and s' are compatible, a contradiction). Therefore, S is countable and $\mathcal{B}$

- Let $\mathcal{O} = \{O_n : n \in \omega\}$ be a family of dense open subsets of $2^{\mathbb{R}}$. Then, $\forall n \in \omega$, $D_n := \{p \in P : [p] \subseteq O_n\}$ is dense in P. Hence, $i[D_n]$ is dense in $\mathcal{B} \setminus \{\emptyset\}$ for all $n \in \omega$. By MA($\aleph_0$) on $\mathcal{B}$, there is a filter G of $\mathcal{B}$ such that $G \cap i[D_n] \neq \emptyset$ for each $n \in \omega$. Then (by the lemma) $H = i^{-1}(G) = \{p \in P : i(p) = N_p \in G\}$ and clearly $H \cap D_n \neq \emptyset$ for each $n \in \omega$.

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  Furthermore, H is a filter of P: Since $p \leq q \rightarrow i(p) \leq i(q)$ and G is a filter of $\mathcal{B}$, it follows that H is upward closed. Now, let $p, q \in H$. Then $i(p) = N_p$ and $i(q) = N_q$ are in G; hence $N_p \cap N_q \in G$. However, $N_p \cap N_q = N_{p \cup q}$, and hence $i(p \cup q) \in G$, so $p \cup q \in H$ and clearly $p \cup q \leq p$ and $p \cup q \leq q$. Thus, H is a filter of P. It follows that $\bigcup H$ is a function with $\mathrm{dom}(\bigcup H) \subseteq \mathbb{R}$ and $\mathrm{ran}(\bigcup H) \subseteq 2$. So, extending $\bigcup H$ to a function $f \in 2^{\mathbb{R}}$, we obtain that $f \in \bigcap \mathcal{O}$.
  $\square$

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- The weaker result "$\mathrm{MA}(\aleph_0) \Rightarrow 2^{\mathbb{R}}$ is Baire" has a much easier proof than the one for the previous theorem, and its **keypoint is the** ZF **fact that the poset** $(\mathrm{Fn}(\mathbb{R},2),\supseteq)$ (which is order isomorphic to $(\mathcal{B},\subseteq)$, where $\mathcal{B}$ is the standard base for the Tychonoff topology on $2^{\mathbb{R}}$) **has the c.c.c.**. In fact, its proof readily yields that for any set X,
  - "$(\mathrm{Fn}(X,2),\supseteq)$ **has the c.c.c."** + $\mathrm{MA}(\aleph_0) \Rightarrow 2^X$ **is Baire**.

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Proof Assume $\mathrm{AC}_{\mathrm{fin}}^{\aleph_0}$ and let X be an infinite set. Let S be an antichain in $(\mathrm{Fn}(X,2),\supseteq)$. For each $n \in \omega$, let $S_n = \{p \in S : |p| = n\}$. It is fairly easy to see that since S is an antichain and $\forall s \in S$, $\mathrm{ran}(s) \subseteq 2$, we have that S_n is a finite set for each $n \in \omega$. By $\mathrm{AC}_{\mathrm{fin}}^{\aleph_0}$, it follows that $S = \bigcup_{n\in\omega} S_n$ is countable.

Theorem

$\mathrm{AC}_{\mathrm{fin}}^{\aleph_0} \Leftrightarrow$ *for every infinite set* X, $(\mathrm{Fn}(X,2),\supseteq)$ *has the c.c.c..*

Proof Assume $\mathrm{AC}_{\mathrm{fin}}^{\aleph_0}$ and let X be an infinite set. Let S be an antichain in $(\mathrm{Fn}(X,2),\supseteq)$. For each $n\in\omega$, let $S_n=\{p\in S:|p|=n\}$. It is fairly easy to see that since S is an antichain and $\forall s\in S$, $\mathrm{ran}(s)\subseteq 2$, we have that S_n is a finite set for each $n\in\omega$. By $\mathrm{AC}_{\mathrm{fin}}^{\aleph_0}$, it follows that $S=\bigcup_{n\in\omega}S_n$ is countable.

Assume that for every infinite set X, $(\mathrm{Fn}(X,2),\supseteq)$ has the c.c.c.. Let $\mathcal{A}=\{A_i:i\in\omega\}$ be a countably infinite family of non-empty finite sets. Without loss of generality, we may assume that $\mathcal{A}$ is disjoint. Let $X=\bigcup\mathcal{A}$. By our hypothesis, $(\mathrm{Fn}(X,2),\supseteq)$ has ccc. Let

$$S_0=\{f\in 2^{A_0}:|f^{-1}(\{1\})|=1\},$$

and for $i\in\omega\setminus\{0\}$, let

$$S_i=\{f\in 2^{A_0\cup\cdots\cup A_i}:[f\restriction(A_0\cup\cdots\cup A_{i-1})\equiv\mathbf{0}]\wedge[|f^{-1}(\{1\})\cap A_i|=1]\}.$$

Then $S = \bigcup_{i \in \omega} S_i$ is an antichain in $(\mathrm{Fn}(X, 2), \leq)$, and thus S is countable, and clearly $|S| = \aleph_0$. Let $S = \{s_n : n \in \omega\}$ be an enumeration of S. For $j \in \omega$, let $n_j = \min\{n \in \omega : s_n \in S_j\}$ and $c_j =$ the unique element x of A_j such that $s_{n_j}(x) = 1$. Then $f = \{(j, c_j) : j \in \omega\}$ is a choice function of the family $\mathcal{A}$. □

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Corollary

BPI $\Rightarrow$ *"for every infinite set X, $(\mathrm{Fn}(X, 2), \supseteq)$ has the c.c.c.".*
The implication is not reversible in ZF.

Theorem

MA($\aleph_0$) *restricted to complete Boolean algebras is false in the Second Fraenkel Model of* ZFA. *Thus,* MC $\not\Rightarrow$ (MA($\aleph_0$) *restricted to complete Boolean algebras*) *in* ZFA *set theory, and consequently* MC *(and hence "every compact T_2 space is Baire") does not imply* MA($\aleph_0$) *in* ZFA.

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Proof The set of atoms $A = \bigcup\{A_n : n \in \omega\}$ is a countable disjoint union of pairs $A_n = \{a_n, b_n\}$, $n \in \omega$. Let G be the group of all permutations of A, which fix A_n for each $n \in \omega$. Let Γ be the finite support filter. Then the Second Fraenkel Model $\mathcal{N}$ is the FM model which is determined by M, G, and Γ.

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- The (countable) family $\mathcal{A} = \{A_n : n \in \omega\}$ has no partial choice function in $\mathcal{N}$.
- Let $P = \{f : f$ is a choice function of $\{A_i : i \leq n\}$ for some $n \in \omega\}$, and for $f, g \in P$, declare $f \leq g$ if and only if $f \supseteq g$. Then $(P, \leq) \in \mathcal{N}$ and every antichain in P is finite.

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- Let $P = \{f : f$ is a choice function of $\{A_i : i \leq n\}$ for some $n \in \omega\}$, and for $f, g \in P$, declare $f \leq g$ if and only if $f \supseteq g$. Then $(P, \leq) \in \mathcal{N}$ and every antichain in P is finite.
- The mapping $i : P \to \mathrm{ro}(P) \setminus \{\emptyset\}$ is $i(p) = N_p$.

- The complete Boolean algebra $(\mathrm{ro}(P), \subseteq)$ has the c.c.c.; in fact, every antichain in $\mathrm{ro}(P)$ is finite: Let S be an antichain in B. For every $s \in S$, let

$$W_s = \{p \in P : |p| = n_s \text{ and } i(p) \subseteq s\}$$

where n_s is the least integer n such that there is a $p \in P$ with $i(p) \subseteq s$. Then $W = \bigcup\{W_s : s \in S\}$ is an antichain in P, thus it is finite. Hence, S is finite.

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- $\forall n \in \omega$, the set

$$D_n = \{f \in P : A_n \in \mathrm{dom}(f)\}$$

is dense in P, and hence $i[D_n]$ is dense in $\mathrm{ro}(P)$ for all $n \in \omega$. Let $\mathcal{D} = \{D_n : n \in \omega\}$. If G were an $i[\mathcal{D}]$-generic filter of $\mathrm{ro}(P)$, then $H = i^{-1}(G)$ would be a $\mathcal{D}$-generic filter of P, so $\bigcup H$ would be a choice function of $\mathcal{A}$, which is impossible. Thus, $\mathrm{MA}(\aleph_0)$ is false for the c.c.c. complete Boolean algebra $\mathrm{ro}(P)$. □

Theorem

MA($\aleph_0$) *is false in Mostowski's Linearly Ordered Model of* ZFA. *Thus (by Pincus' transfer theorems),* BPI + *Countable Union Theorem* (CUT) $\not\Rightarrow$ MA($\aleph_0$) *in* ZF.

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Proof Start with a ground model M with a linearly ordered set $(A, \preceq)$ of atoms which is order isomorphic to $(\mathbb{Q}, \leq)$. G is the group of all order atutomorphims of $(A, \preceq)$ and Γ is finite support filter. The Mostowski model $\mathcal{N}$ is the model determined by M, G and Γ.

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The power set of the set A of atoms in Mostowski's model is Dedekind-finite, and hence $\forall X$ (2^X is Baire) is false in $\mathcal{N}$. Since BPI is true in $\mathcal{N}$, it follows that $\forall X$ (($\mathrm{Fn}(X, 2), \supseteq$) has the c.c.c.) is also true in $\mathcal{N}$. Thus, $\mathsf{MA}(\aleph_0)$ is false in Mostowski's model. □

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Proof Start with a ground model M of ZFA + AC with a countable set A of atoms. Let G be the group of all permutations of A and let Γ be the finite support filter. Then the Basic Fraenkel Model $\mathcal{N}$ is the permutation model determined by M, G and Γ.

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MA($\aleph_0$) is false in $\mathcal{N}$, since $\forall X$, $\mathrm{Fn}(X, 2)$ has the c.c.c. (for $\mathrm{AC}_{\mathrm{fin}}^{\aleph_0}$ is true in $\mathcal{N}$) and A is amorphous (where A is the set of atoms), and hence 2^A is not Baire in $\mathcal{N}$. □

Theorem

If ZFA *is consistent, so is* ZFA + MA^* + $\neg\mathrm{MA}(\aleph_0)$ + (DF = F) + CUT.

(DF = F stands for "every Dedekind-finite is finite" and CUT is the Countable Union Theorem.)

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Proof Start with a model M of ZFA + AC + CH, in which there is a set of atoms $A = \bigcup\{A_n : n \in \omega\}$ which is a countable disjoint union of $\aleph_1$-sized sets. Let G be the group of all permutations of A, which fix A_n for every $n \in \omega$. Let Γ be the (normal) filter of subgroups of G generated by $\{\mathrm{fix}_G(E) : E = \bigcup_{i \in I} A_i,\ I \in [\omega]^{<\omega}\}$. Let $\mathcal{N}$ be the FM model determined by M, G and Γ.

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- In $\mathcal{N}$, CH is true, hence so is MA^*.

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- In $\mathcal{N}$, CH is true, hence so is MA*.
- The family $\mathcal{A} = \{A_n : n \in \omega\}$ has no partial Kinna–Wagner Selection function in $\mathcal{N}$.

- DF = F is true in $\mathcal{N}$, and hence $\forall X$, $(\mathrm{Fn}(X,2),\supseteq)$ has the c.c.c.:
  Let $x \in \mathcal{N}$ be a non-well-orderable set and let $E = \bigcup\{A_i : i \leq k\}$ be a support of x. Then there exists an element $z \in x$ and a $\phi \in \mathrm{fix}_G(E)$ such that $\phi(z) \neq z$. Let E_z be a support of z; wlog assume that $E_z = E \cup A_{k+1}$ and that $\phi \in \mathrm{fix}_G(A \setminus A_{k+1})$. Let

$$y = \{\psi(z) : \psi \in \mathrm{fix}_G(A \setminus A_{k+1})\}.$$

  Then y is well-orderable and infinite; otherwise the index of the proper subgroup

$$H = \{\eta \in \mathrm{fix}_G(A \setminus A_{k+1}) : \eta(z) = z\}$$

  in $\mathrm{fix}_G(A \setminus A_{k+1})$ is finite. However, $\mathrm{fix}_G(A \setminus A_{k+1})$ is isomorphic to $\mathrm{Sym}(\aleph_1)$, and by a result of Gaughan, every proper subgroup of $\mathrm{Sym}(\aleph_1)$ has uncountable index. We have reached a contradiction, and thus y is infinite.

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- MA($\aleph_0$) is false in $\mathcal{N}$: **(a)** $(\mathrm{Fn}(A,2),\supseteq)$ has the c.c.c. (since by CUT in $\mathcal{N}$ – or DF = F –, it follows that $\mathrm{AC}^{\aleph_0}_{\mathrm{fin}}$ is also true in $\mathcal{N}$) **(b)** $\mathcal{A} = \{A_n : n \in \omega\}$ has no partial Kinna–Wagner selection function in $\mathcal{N}$, and hence 2^A $(= 2^{\bigcup \mathcal{A}})$ is not Baire in $\mathcal{N}$. □

- CUT is true in $\mathcal{N}$: Fairly similar argument to the one for DF $=$ F in $\mathcal{N}$.
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Theorem

There is a ZF *model in which* (DF $=$ F) $+$ CUT *is true, whereas* MA($\aleph_0$) *is false.*

- CUT is true in $\mathcal{N}$: Fairly similar argument to the one for DF $=$ F in $\mathcal{N}$.
- MA($\aleph_0$) is false in $\mathcal{N}$: **(a)** $(\mathrm{Fn}(A,2),\supseteq)$ has the c.c.c. (since by CUT in $\mathcal{N}$ – or DF $=$ F –, it follows that $\mathrm{AC}_{\mathrm{fin}}^{\aleph_0}$ is also true in $\mathcal{N}$) **(b)** $\mathcal{A} = \{A_n : n \in \omega\}$ has no partial Kinna–Wagner selection function in $\mathcal{N}$, and hence 2^A $(= 2^{\bigcup \mathcal{A}})$ is not Baire in $\mathcal{N}$. □

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There is a ZF *model in which* (DF $=$ F) $+$ CUT *is true, whereas* MA($\aleph_0$) *is false.*

Proof This follows from the facts that $\Phi = (\mathrm{DF} = \mathrm{F}) + \mathrm{CUT} + \neg\mathrm{MA}(\aleph_0)$ is a conjunction of injectively boundable statements and Φ has a ZFA model, so by Pincus' transfer theorems it follows that Φ has a ZF model. □

Theorem

If ZF *is consistent, then so is* ZF + $\mathrm{MA}^* + \neg\mathrm{AC}^{\aleph_0}$.

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Proof We start with a countable transitive model M of ZF $+$ CH, and we extend M to a symmetric model N of ZF with the same reals as in M, but which does not satisfy AC$^{\aleph_0}$.

Theorem

If ZF *is consistent, then so is* ZF + MA* + $\neg AC^{\aleph_0}$.

Proof We start with a countable transitive model M of ZF + CH, and we extend M to a symmetric model N of ZF with the same reals as in M, but which does not satisfy $AC^{\aleph_0}$.

Let $P = \mathrm{Fn}(\omega \times \aleph_1 \times \aleph_1, 2, \aleph_1)$ be the set of all partial functions p with $|p| < \aleph_1$, $\mathrm{dom}(p) \subset \omega \times \aleph_1 \times \aleph_1$ and $\mathrm{ran}(p) \subseteq 2$, partially ordered by reverse inclusion, i.e., $p \leq q$ if and only if $p \supseteq q$. Since $\aleph_1$ is a regular cardinal, it follows that $(P, \leq)$ is a $\aleph_1$-closed poset. Hence, forcing with P adds only new subsets of $\aleph_1$ and no new subsets of cardinals $< \aleph_1$. Therefore, forcing with P adds no new reals; it only adds new subsets of $\mathbb{R}$.

Let $a_{n,m} = \{j \in \aleph_1 : \exists p \in G, p(n,m,j) = 1\}$, $n \in \omega$, $m \in \aleph_1$, let $A_n = \{a_{n,m} : m \in \aleph_1\}$, $n \in \omega$, and let $\mathcal{A} = \{A_n : n \in \omega\}$.
Every permutation ϕ of $\omega \times \aleph_1$ induces an order-automorphism of $(P, \leq)$ by requiring for every $p \in P$,

$$\begin{aligned} \operatorname{dom} \phi(p) &= \{(\phi(n,m),k) : (n,m,k) \in \operatorname{dom}(p)\}, \\ \phi(p)(\phi(n,m),k) &= p(n,m,k). \end{aligned}$$

Let $\mathcal{G}$ be the group of all order-automorphisms of $(P, \leq)$ induced (as above) by all those permutations ϕ of $\omega \times \aleph_1$, which satisfy

$$\phi(n,m) = (n,m') \text{ for all ordered pairs } (n,m) \in \omega \times \aleph_1.$$

(So ϕ is essentially such that $\forall n \in \omega$, $\exists$ permutation ϕ_n of $\aleph_1$ so that $\phi(n,m) = (n, \phi_n(m))$ for all $n \in \omega$. Further, the effect of ϕ on a condition $p \in P$ is that ϕ changes only the second coordinate of p.)

For every finite subset $E \subset \omega \times \aleph_1$, let $\mathrm{fix}_{\mathcal{G}}(E) = \{\phi \in \mathcal{G} : \forall e \in E, \phi(e) = e\}$ and let Γ be the filter of subgroups of $\mathcal{G}$ generated by $\{\mathrm{fix}_{\mathcal{G}}(E) : E \subset \omega \times \aleph_1, |E| < \aleph_0\}$. An element $x \in M$ is called *symmetric* if there exists a finite subset $E \subset \omega \times \aleph_1$ such that $\forall \phi \in \mathrm{fix}_{\mathcal{G}}(E), \phi(x) = x$. Under these circumstances, we call E a *support* of x. An element $x \in M$ is called *hereditarily symmetric* if x and every element of the transitive closure of x is symmetric. Let HS be the set of all hereditarily symmetric names in M and let

$N = \{\tau_G : \tau \in \mathrm{HS}\} \subset M[G]$

be the symmetric extension model of M.

Since M and N have the same reals, we have MA^* is true in the model N.

Furthermore, the countable family $\mathcal{A} = \{A_n : n \in \omega\}$ has no choice function, and thus $\mathrm{AC}^{\aleph_0}$ is false in N. □

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Thank You!