On the Ellis–Numakura Lemma, Free Idempotent Ultrafilters on ω and Choice

ELEFTHERIOS TACHTSIS

Department of Statistics and Actuarial-Financial Mathematics
University of the Aegean
Karlovassi 83200, Samos, Greece

2023 International Conference on Topology and its Applications

July 03–07, 2023

Nafpaktos, Greece

Overview

I will discuss, in set theory without the full power of the Axiom of Choice (**AC**), the deductive strength of the following two statements:

1. The **Ellis–Numakura Lemma**: "*Every compact Hausdorff right topological semigroup has an idempotent element*".
2. "*There exists a free idempotent ultrafilter on* ω".

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2. "*There exists a free idempotent ultrafilter on* ω".

Why is it important to study their mutual relationship as well as their possible interrelations with **AC** and weak forms of **AC**?

A **chief reason** is that both of the aforementioned results are (known to be) strongly related to **Hindman's Theorem**:
"*For any finite colouring of* ω, *there exists an infinite set* $H \subseteq \omega$ *such that the set*

$$\mathrm{FS}(H) = \{\sum_{x \in F} x : F \in [H]^{<\omega} \setminus \{\emptyset\}\}$$

is monochromatic"
(initially known as the "**Graham–Rothschild conjecture**" and proven to be true by Hindman in 1974), which is a **cornerstone of the Ramsey theory of numbers** and also **provable in ZF**, as shown by W. W. Comfort in 1977.

Definition

A *semigroup* is a non-empty set S together with a mapping $(x, y) \mapsto xy$ of $S \times S$ to S such that $x(yz) = (xy)z$ for all $x, y, z \in S$. In other words, a semigroup is a non-empty set with an associative binary operation. If, in addition, the binary operation is commutative, then the semigroup is called *abelian semigroup*.

Let S be a semigroup. For a fixed element $s \in S$, the mapping $x \mapsto xs$ of S to S is called the *right action* of s on S, and is denoted by ρ_s.

An element s of a semigroup S is called *idempotent* if $ss = s$.

A non-empty subset H of a semigroup S is called a *subsemigroup* of S if $xy \in H$ for all $x, y \in H$.

A *right topological semigroup* is a semigroup S with a topology $\mathcal{T}$ on S such that for all $s \in S$, the right action ρ_s of s on S is a continuous mapping of S to itself.

Theorem

(Ellis–Numakura Lemma (ENL)) *Every compact Hausdorff right topological semigroup has an idempotent element.*

It is part of the folklore that

$$\textbf{Zorn's Lemma}\ (\Longleftrightarrow \textbf{AC}) \Rightarrow \textbf{ENL}.$$

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Beyond that, and prior to [T2018], no other proofs for **ENL** were known (to the best of our knowledge), and thus resulting in a gap in information about the set-theoretic strength of **ENL**. For example, the **first** of the subsequent two (intriguing) questions is **still open**, while the **second** has been **completely settled** in [T2018] (**in the negative**).

- Is **ENL** provable in **ZF**?

- Is **ENL** equivalent to **AC**?

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PW: *The power set of a well orderable set can be well ordered*.

In **ZF**, **PW** $\iff$ **AC**, but in **ZFA**, **PW** $\not\Rightarrow$ **AC**.

Theorem

*In **ZF**, **PW** $\Rightarrow$ **ENL**.*

Deductive strength of **ENL**

- Boolean Prime Ideal Theorem (**BPI**): *Every Boolean algebra has a prime ideal*. Equivalently, *every proper filter on a set can be extended to an ultrafilter*.
- Multiple Choice Axiom (**MC**): *Every family of non-empty sets has a multiple choice function* (i.e. a function which chooses from every member of the family a nonempty finite subset).
- **LW**: *Every linearly ordered set can be well ordered*.
- $\mathbf{AC_{fin}}$: *Every family of nonempty finite sets has a choice function*.

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- **LW**: *Every linearly ordered set can be well ordered.*
- $\mathbf{AC_{fin}}$: *Every family of nonempty finite sets has a choice function.*

1. In **ZF**, $\mathbf{AC_{fin}}$ is strictly weaker than **BPI**, which in turn is strictly weaker than **AC**,
2. In **ZF**, **MC** $\iff$ **LW** $\iff$ **AC**, but in **ZFA**, **MC** $\not\Rightarrow$ $\mathbf{AC_{fin}}$ (e.g. Second Fraenkel Model) and **LW** $\not\Rightarrow$ **MC** (e.g. Basic Fraenkel Model).

A short list of results to be discussed:

1. **ENL** for well orderable semigroups is provable in **ZF**.
2. **ENL** for Loeb semigroups is provable in **ZF** (where a space $\mathbf{X} = (X, \tau)$ is called *Loeb* if the family of all nonempty closed sets in $\mathbf{X}$ has a choice function).
3. $\mathbf{BPI} \Rightarrow \mathbf{MENL} \Rightarrow \mathbf{ENL} + \mathbf{AC_{fin}}$, where **MENL** is "For every family $\mathcal{A} = \{(S_i, \cdot_i, \mathcal{T}_i) : i \in I\}$ of non-trivial compact Hausdorff right topological semigroups, there exists a function f with domain I such that $f(i)$ is an idempotent of S_i, for all $i \in I$". Hence, $\mathbf{ENL} \not\Rightarrow \mathbf{AC}$ in **ZF**.
4. In **ZFA**, $\mathbf{MC} \Rightarrow \mathbf{ENL_{abel}}$, i.e. **ENL** for abelian semigroups.
5. In **ZFA**, $\mathbf{LW} \Rightarrow \mathbf{ENL_{lo}}$, i.e. **ENL** for linearly ordered semigroups.
6. $\mathbf{ENL_p} \not\Rightarrow \mathbf{BPI}$ in **ZF**, where $\mathbf{p} \in \{\mathbf{abel}, \mathbf{lo}\}$, and $\mathbf{ENL} \not\Rightarrow \mathbf{P}$ in **ZFA**, where $\mathbf{P} \in \{\mathbf{MC}, \mathbf{LW}\}$.

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Claim. For every closed subsemigroup A of S there is a definable way to choose an $a \in A$ such that $C := \{x \in A : xa = a\} \neq \emptyset$. (Note that C is a closed subsemigroup of S.)

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Proof of claim. Using $\leq$, a and C will be defined via transfinite recursion.

Put $R_0 = A$. As $R_0 \neq \emptyset$ (semigroups are non-empty by definition), pick any $r_0 \in R_0$. If $r_0 \in R_0 r_0$, then we are done. If $r_0 \notin R_0 r_0$, let

$$R_1 = R_0 r_0.$$

Then R_1 is a closed, proper subsemigroup of R_0. Since $\leq$ is a well ordering of S, let

$$r_1 = \min(R_1).$$

If $r_1 \in R_1 r_1$, then we are done; otherwise, $R_1 r_1$ is a closed, proper subsemigroup of S. Let $R_2 := R_1 r_1$ and continue as in the first step.

Suppose that α is a limit ordinal and that we have constructed a descending sequence $(R_\beta)_{\beta<\alpha}$ of closed subsemigroups of S (hence, each one of them is contained in $A = R_0$) and a sequence $(r_\beta)_{\beta<\alpha}$ such that $\forall\beta < \alpha$, $r_\beta \in R_\beta$.
By compactness of S, we deduce that

$$R_\alpha := \bigcap_{\beta<\alpha} R_\beta \neq \emptyset.$$

Furthermore, R_α is a closed subsemigroup of S. Let

$$r_\alpha := \min(R_\alpha).$$

At a successor ordinal stage we work exactly as in the first two steps of the recursion.
Since Ord (the class of all ordinal numbers) is proper, the recursion must terminate at some ordinal stage κ. This means that at stage κ, we have ended up with a closed subsemigroup R_κ of A and with an $r_\kappa \in R_\kappa r_\kappa$. Letting $a := r_\kappa$ and $C := \{x \in A : xa = a\}$, the claim is proved. $\square$**(Claim)**

Via recursion, and using the claim, we find an idempotent of S: By the claim (applied to S), $\exists w_0 \in S$ such that $C_0 = \{x \in S : xw_0 = w_0\} \neq \emptyset$. If $w_0 \in C_0$, we are done. Otherwise, by (the proof of) the claim applied to C_0, we find a $w_1 \in C_0$ such that $C_1 = \{x \in C_0 : xw_1 = w_1\} \neq \emptyset$. If $w_1 \in C_1$, we are done. Otherwise, we apply the claim to C_1.

At a limit ordinal stage, we take the intersection of the descending family of closed subsemigroups of S constructed in the previous stages (the intersection is nonempty due to compactness of S), and we apply the claim to this intersection.

At a successor ordinal stage, we work as in the first two steps of the recursion.

As Ord is a proper class, at some ordinal stage we have obtained a closed subsemigroup A_κ of S and a $w_\kappa \in A_\kappa$ such that $w_\kappa \in C_\kappa = \{x \in A_\kappa : xw_\kappa = w_\kappa\}$. Thus $w_\kappa w_\kappa = w_\kappa$. $\square$

Corollary

In **ZFA**, **LW** *(every linearly ordered set can be well ordered) implies* **ENL** *for linearly ordered semigroups.*

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Recall that, **LW** is equivalent to **AC** in **ZF**, but it is **not** equivalent to **AC** in **ZFA**.

The arguments of the previous proof glaringly show that to establish **ENL** it is **sufficient to assume the existence of choice functions for the family of all nonempty closed subsets of the semigroup**, i.e. to assume that the semigroup is a Loeb space.

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We thus immediately obtain the following result.

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ENL *for Loeb semigroups is provable in* **ZF***.*

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We thus immediately obtain the following result.

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ENL *for Loeb semigroups is provable in* **ZF**.

This, together with the fact that **BPI implies "Every compact Hausdorff space is Loeb", but does not imply AC in ZF**, yields the following theorem which sheds further light on the deductive strength of **ENL**.

Theorem

BPI *implies* **ENL**. *Thus* **ENL** *is not equivalent to* **AC** *in* **ZF**.

It is **unknown** whether or not **ENL** is equivalent to **BPI**. However, as we will see later on, certain fragments of **ENL** are *not* equivalent to **BPI** (and recall also **ENL** restricted to well-orderable, or Loeb, semigroups).

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Theorem

$\mathbf{BPI} \Rightarrow \mathbf{MENL} \Rightarrow \mathbf{ENL} + \mathbf{AC}_{\mathbf{fin}}$.

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Theorem

BPI $\Rightarrow$ **MENL** $\Rightarrow$ **ENL** $+$ $\mathbf{AC_{fin}}$.

Proof. We first recall the following two facts about **BPI**:

1. **BPI** $\iff$ "products of compact Hausdorff spaces are compact".
2. **BPI** $\Rightarrow$ "products of nonempty compact Hausdorff spaces are nonempty" $\Rightarrow$ "compact Hausdorff spaces are Loeb".

BPI $\Rightarrow$ **MENL**: Let $\mathcal{A} = \{(S_i, \cdot_i, \mathcal{T}_i) : i \in I\}$ be a family of non-trivial compact Hausdorff right topological semigroups and also let $A = \prod_{i \in I} S_i$. By (2) above, $A \neq \emptyset$, and by (1) and (2), A with the product topology is a compact Hausdorff Loeb space.
We define a binary operation $\cdot$ on A by requiring

$$(\forall f \in A)(\forall g \in A)(\forall i \in I)[(f \cdot g)(i) = f(i) \cdot_i g(i)].$$

Then $(A, \cdot)$ is a right topological semigroup. As **BPI** implies **ENL**, A has an idempotent element, h say. Then, for every $i \in I$, $h(i)$ is an idempotent of S_i.
MENL $\Rightarrow$ **ENL**: Straightforward.

MENL $\Rightarrow$ **AC**$_{\text{fin}}$: Let $\mathcal{A} = \{S_i : i \in I\}$ be a family of non-empty finite sets.
For each $i \in I$, we let $\mathcal{T}_i$ be the discrete topology on S_i and we also define a binary operation $\cdot_i$ on S_i by requiring

$$(\forall x \in S_i)(\forall y \in S_i)(x \cdot_i y = y).$$

Then, for each $i \in I$, $(S_i, \cdot_i, \mathcal{T}_i)$ is a compact Hausdorff right topological semigroup. By **MENL**, there exists a function f with domain I such that, for every $i \in I$, $f(i)$ is an idempotent element of S_i. Clearly, f is a choice function for the family $\mathcal{A}$. Thus AC$_{\text{fin}}$ is true. $\square$

Theorem

In **ZFA**, **MC** $\Rightarrow$ **ENL** *for abelian semigroups.*

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Proof. Let S be a compact Hausdorff right topological abelian semigroup. Let f be a multiple choice function for the family of all nonempty closed subsets of S. As in the proof of **ENL** for well-orderable (or Loeb) semigroups, it suffices to show the following:

Claim. For every closed subsemigroup A of S there is a definable way to choose an $a \in A$ such that the closed subsemigroup $\{x \in A : xa = a\}$ is nonempty.

Proof of the claim. Follow the proof of the corresponding claim for **ENL** restricted to well-orderable semigroups, but at each ordinal stage α of the transfinite recursion, define

$$r_\alpha = \prod_{x \in f(R_\alpha)} x.$$

Recall that $f(R_\alpha)$ is (nonempty) finite and note that r_α is well-defined since S is abelian. □

Theorem

1. *If* $P \in \{\mathbf{LW}, \mathbf{MC}\}$, *then* $\mathbf{MENL} \not\Rightarrow P$ *in* **ZF** *(or in* **ZFA***).*
2. *For every* $P \in \{\mathbf{ENL_{abel}}, \mathbf{ENL_{lo}}\}$, $P \not\Rightarrow$ **BPI** *in* **ZFA***.*
3. $\mathbf{MC} \not\Rightarrow \mathbf{MENL}$ *in* **ZFA***.*

Theorem

1. *If* $P \in \{\mathbf{LW}, \mathbf{MC}\}$*, then* $\mathbf{MENL} \not\Rightarrow P$ *in* **ZF** *(or in* **ZFA***).*
2. *For every* $P \in \{\mathbf{ENL_{abel}}, \mathbf{ENL_{lo}}\}$*,* $P \not\Rightarrow \mathbf{BPI}$ *in* **ZFA***.*
3. $\mathbf{MC} \not\Rightarrow \mathbf{MENL}$ *in* **ZFA***.*

The above result follows from the previous theorems and the following known facts:

- **BPI** implies neither **LW** nor **MC** in **ZF** (or in **ZFA**).
- **MC** does not imply $\mathbf{AC_{fin}}$, and thus the stronger **BPI**, in **ZFA**.

Free idempotent ultrafilters on ω

Let X be an infinite set. βX denotes the *Stone space* of the Boolean algebra $\mathcal{B} = (\wp(X), \cap, \cup, X, \emptyset)$, i.e., βX is the set of all ultrafilters on X together with the topology $\mathcal{T}$ which has as a base the set of all (clopen) sets of the form

$$[A] = \{\mathcal{U} \in \beta X : A \in \mathcal{U}\},\ A \in \wp(X).$$

$(\beta X, \mathcal{T})$ is Hausdorff and all principal ultrafilters $\langle x \rangle$ $(x \in X)$ are isolated points. X with the discrete topology embeds as an open relatively discrete subspace of βX via $x \mapsto \langle x \rangle$.

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$(\beta X, \mathcal{T})$ is Hausdorff and all principal ultrafilters $\langle x \rangle$ $(x \in X)$ are isolated points. X with the discrete topology embeds as an open relatively discrete subspace of βX via $x \mapsto \langle x \rangle$.

Definition

(Glazer) Let $\mathcal{U}, \mathcal{V} \in \beta\omega$. We define

$$\mathcal{U} + \mathcal{V} = \{A \subseteq \omega : \{n \in \omega : A - n \in \mathcal{V}\} \in \mathcal{U}\},$$

where $A - n = \{x \in \omega : x + n \in A\}$.

The addition '+' on $\beta\omega$ extends the usual addition on ω since $\langle n \rangle + \langle m \rangle = \langle n + m \rangle$, and is known to be non-commutative.

- It was Glazer who first proved the existence of a free idempotent ultrafilter on ω, i.e. of a free ultrafilter $\mathcal{U}$ on ω such that $\mathcal{U} + \mathcal{U} = \mathcal{U}$.

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The following two facts are part of the folklore.

Fact

*(**ZF**) The space $(\beta\omega, +)$ is a right topological semigroup.*

Fact

*(**ZF**) If $(\beta\omega \setminus \omega, +)$ is nonempty, then $(\beta\omega \setminus \omega, +)$ is a closed right topological subsemigroup of $(\beta\omega, +)$.*

- **BPI**$_\omega$: *Every proper filter on* ω *can be extended to an ultrafilter on* ω.
- **BPI**$_\mathbb{R}$: *Every proper filter on* $\mathbb{R}$ *can be extended to an ultrafilter on* $\mathbb{R}$.

Fact

BPI$_\omega$ $\iff$ *"the Cantor cube* $2^\mathbb{R}$ *is compact".*

Fact

BPI$_\mathbb{R}$ $\iff$ *"the Cantor cube* $2^{\wp(\mathbb{R})}$ *is compact"* $\Rightarrow$ **BPI**$_\omega$ + *"*$2^\mathbb{R}$ *is Loeb". The latter implication is not reversible in* **ZF**.

Fact

*(**ZF**) For every infinite set X, βX embeds as a closed subspace of $2^{\wp(X)}$. Thus, $\beta\omega$ embeds as a closed subspace of $2^{\mathbb{R}}$.*

Theorem

The following hold:
(i) $\mathbf{BPI}_{\mathbb{R}} \Rightarrow \mathbf{BPI}_{\omega}$ + *"$2^{\mathbb{R}}$ is Loeb"* $\Rightarrow$ *"there exists a free idempotent ultrafilter on ω".*
(ii) **ENL** + $\mathbf{BPI}_{\omega} \Rightarrow$ *"there exists a free idempotent ultrafilter on ω".*
(iii) "There exists a free idempotent ultrafilter on ω" $\not\Rightarrow \mathbf{BPI}_{\omega}$ *in* **ZF**. *Hence, it is relatively consistent with* **ZF** *that there exists a free idempotent ultrafilter on ω but $\beta\omega$ is not compact.*

Definition

Let $H \subseteq \omega$. We define

$$\mathrm{FS}(H) = \left\{ \sum_{x \in F} x : F \in [H]^{<\omega} \setminus \{\emptyset\} \right\}.$$

Theorem

(Hindman's Theorem) *If ω is partitioned into finitely many pieces then one of the pieces, A, contains an infinite set H such that $\mathrm{FS}(H) \subseteq A$. (Such a set A is called an IP set.)*

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- Comfort in 1977 showed that *Hindman's theorem is provable in* **ZF**. (Actually, Comfort credits the argument to Baumgartner.)
- The standard proof of Hindman's theorem is due to Galvin who (based on Glazer's result) showed that, in **ZF**, *any element of a free idempotent ultrafilter on ω is an IP set*.

Fact

"There exists a free ultrafilter on ω" $\iff$ "every countable filter base on ω is included in an ultrafilter on ω".

Definition

If $x, y \in \omega$, $x \neq y$, then $x, y, x+y$ is called a *Schur triple*.

Theorem

The following are pairwise equivalent:
(i) There exists a free ultrafilter on ω.
(ii) There exists a free ultrafilter on ω which is not idempotent.
(iii) For every IP set $A \subseteq \omega$ there exists a free ultrafilter $\mathcal{F}$ on ω such that $A \in \mathcal{F}$.

Proof. (i) $\to$ (ii) We first prove (via mathematical induction) that there exists an infinite subset A of ω which contains no Schur triple.
Let a_0, a_1 be positive integers with $a_0 < a_1$. Assume that, for some $n \in \omega \setminus \{0\}$, we have chosen natural numbers $a_0, a_1, \ldots, a_n$ such that

$$(\forall j \in \{1, \ldots, n\})(\forall y \in \mathrm{FS}(\{a_k : k < j\})(y < a_j).$$

(In particular, note that $a_0 < a_1 < \cdots < a_n$.)
As $\mathrm{FS}(\{a_0, \ldots, a_n\})$ is finite, we may let a_{n+1} be the least natural number which is greater than every element of $\mathrm{FS}(\{a_0, \ldots, a_n\})$. This concludes the inductive step. Put

$$A = \{a_n : n \in \omega\}.$$

By the above construction, A is infinite and has no Schur triples.
As $|A| = \aleph_0$, there exists (by hypothesis) a free ultrafilter on A, $\mathcal{F}$ say. Let $\mathcal{G}$ be the filter on ω which is generated by $\mathcal{F}$.
As $\mathcal{F}$ is a free ultrafilter on A and $\mathcal{G}$ is generated by $\mathcal{F}$, $\mathcal{G}$ is a free ultrafilter on ω such that $A \in \mathcal{G}$.
By Galvin's result, $\mathcal{G}$ is not idempotent because, since $A \in \mathcal{G}$ has no Schur triples, there is no infinite set $H \subseteq A$ such that $\mathrm{FS}(H) \subseteq A$.

(ii) $\to$ (i) This is straightforward.

(i) $\to$ (iii) Let $A \subseteq \omega$ be an IP set, then A contains an infinite set H with $\mathrm{FS}(H) \subseteq A$. Let

$$B = \{\mathrm{FS}(H \setminus Q) : Q \in [H]^{<\omega}\}.$$

Since $|H| = \aleph_0$ and $|[\omega]^{<\omega}| = \aleph_0$, it follows that $|B| = \aleph_0$. Furthermore, B is a filter base such that $\bigcap B = \emptyset$. Thus, by our hypothesis and the previous Fact, there is a free ultrafilter $\mathcal{U}$ on ω such that $B \subseteq \mathcal{U}$. Since $\mathrm{FS}(H) \in \mathcal{U}$ and $\mathrm{FS}(H) \subseteq A$, we obtain $A \in \mathcal{U}$.

(iii) $\to$ (i) Since $\mathrm{FS}(\omega) \subseteq \omega$, our hypothesis readily implies that there exists a free ultrafilter on ω. $\square$

Corollary

$\mathbf{BPI}_\omega$ *implies there exists a free ultrafilter on* ω *which is not idempotent. The latter implication is not reversible in* ZF.

Proof. Follows from the previous theorem and a result of Keremedis–Hall–Tachtsis (2013) that "there exists a free ultrafilter on ω" does not imply $\mathbf{BPI}_\omega$ in **ZF** (via the construction of a novel **ZF**-model with the required properties). □

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Problem

*Does either of "there exists a free ultrafilter on ω" and (the strictly stronger in **ZF**) **BPI**$_\omega$ imply "there exists a free idempotent ultrafilter on ω"?*

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Does either of "there exists a free ultrafilter on ω*" and (the strictly stronger in* **ZF***)* **BPI**$_\omega$ *imply "there exists a free idempotent ultrafilter on* ω*"?*

This is **still open**, but a promising **ZF**-model (due to Mathias) towards a (possibly) negative answer is the following:

We start with Feferman's model $\mathcal{M}$ as the ground model in which countably many generic reals are added but no set to collect them, $\mathbf{AC^{WO}}$ (the axiom of choice for well-ordered families) is true, and all ultrafilters on ω are principal. We then force with a suitable notion of forcing $\mathbb{P}$ over $\mathcal{M}$ so that the generic extension $\mathcal{M}[G]$ of $\mathcal{M}$ has a free non-idempotent ultrafilter on ω.

Let $\mathbb{P} = [\omega]^\omega/\mathrm{fin}$. We define a partial order $\leq$ on $\mathbb{P}$ by requiring:

$$[A] \leq [B] \iff A \setminus B \text{ is finite}$$

where for $X \in [\omega]^\omega$, $[X]$ is the equivalence class of X, i.e., $[X] = \{X \triangle Z : Z \in [\omega]^{<\omega}\}$.

A word of caution: Since in Feferman's model $\mathcal{M}$, all ultrafilters on ω are principal, it follows that $\mathbf{AC}_{\mathbb{R}}$ (the axiom of choice for sets of sets of reals) is false in $\mathcal{M}$. Thus there is a problem in choosing simultaneously representatives from the equivalence classes. As a result, the above forcing notion $(\mathbb{P}, \leq)$ should actually be reformulated.

Either take as forcing conditions infinite subsets p, q of ω with the agreement that if $p \triangle q$ is finite, then they force the same statements, and that p is a stronger condition than q if $p \setminus q$ is finite,

or take $\mathbb{P}$ as defined above and define $\leq$ on $\mathbb{P}$ by either "$p \leq q$ iff $\forall x \in p\ \forall y \in q\ (x \setminus y$ is finite)" or "$p \leq q$ iff $\exists x \in p\ \exists y \in q\ (x \setminus y$ is finite)".
Then all the arguments of the proof remain essentially the same. Thus for the sake of simplicity, we keep the original formulation of $(\mathbb{P}, \leq)$.

Fact

$\mathbb{P}$ is $\aleph_1$-closed (i.e., whenever $\gamma < \aleph_1$ and $(p_\xi)_{\xi<\gamma}$ is a decreasing sequence of elements of $\mathbb{P}$ – that is, $\xi < \eta \to p_\eta \leq p_\xi$ – then $(\exists q \in \mathbb{P})$ $(\forall \xi < \gamma)$ $(q \leq p_\xi)$).

Let G be a $\mathbb{P}$-generic set over $\mathcal{M}$ and let $\mathcal{M}[G]$ be the generic extension model of $\mathcal{M}$.

Fact

$\mathcal{M}[G]$ has the same reals as $\mathcal{M}$.

Let

$$\mathcal{U} = \bigcup G.$$

Fact

$\mathcal{U}$ is a free ultrafilter on ω in $\mathcal{M}[G]$.

Fact

$\mathcal{U}$ is not idempotent in $\mathcal{M}[G]$.

Proof. It suffices to show that there exists $A \in \mathcal{U}$ such that A contains no Schur triple. Then, by Galvin's result, we will obtain that $\mathcal{U}$ is not idempotent.

Let

$$D = \{p \in \mathbb{P} : (\exists A \in p)(A \text{ has no Schur triple})\}.$$

Clearly, $D \in \mathcal{M}$. Furthermore, $D \neq \emptyset$ (due to a previous argument).

D is dense in $\mathbb{P}$. Let $p = [X] \in \mathbb{P}$. We will find a $d \in D$ such that $d \leq p$. As X is infinite, we may easily construct (as done previously) an infinite $Y \subseteq X$ which has no Schur triples.

It follows that the set

$$d = [Y]$$

belongs to D. As $Y \subseteq X$, we obtain

$$d = [Y] \leq [X] = p,$$

i.e. D is dense in $\mathbb{P}$ as required.

Since G is a $\mathbb{P}$-generic set over $\mathcal{M}$, we have $G \cap D \neq \emptyset$. Let $g \in G \cap D$. By the definition of D, there exists $A \in g$ such that A has no Schur triples. Furthermore, as $[A] = g \in G$, we have $A \in \mathcal{U}$. Thus, by Galvin's result, the ultrafilter $\mathcal{U}$ is not idempotent. $\square$

Problem

1. *Are all ultrafilters on ω in $\mathcal{M}[G]$ non-idempotent?*

2. *Are the ultrafilters $\mathcal{U}+\mathcal{U}, \mathcal{U}+\mathcal{U}+\mathcal{U}, \ldots$ non-idempotent in $\mathcal{M}[G]$? Are these ultrafilters as well as their $\in$-isomorphic ultrafilters all the ultrafilters on ω in $\mathcal{M}[G]$?*

3. *Is any of* **ENL** *and* **ENL** *for abelian semigroups, and* **ENL** *for linearly orderable semigroups provable in* **ZF**?

4. *Does either of* **MC** *and* **PW** *imply* **ENL** *in* **ZFA**?

5. *Does* **ENL** *imply either of "there exists a free idempotent ultrafilter on ω" and* $\mathbf{BPI}_\omega$?

6. *Does the statement "there exists a free ultrafilter on ω" imply "there exists a free idempotent ultrafilter on ω"?*

7. *Does* $\mathbf{AC}_{\mathbf{fin}}$ *imply* **MENL**? *Does* **MENL** *imply* **BPI**?

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Thank You!