Deterministic Arbitrage

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Basic Payoffs of European Call and Put Options

Notation

Let S_T be the price of the underlying asset at maturity T, and let K be the strike price.

European Call

The payoff of a European call option at maturity is

$$(S_T - K)^+ = \max(S_T - K, 0).$$

Equivalently,

$$(S_T - K)^+ = \begin{cases} 0, & S_T \leq K, \\ S_T - K, & S_T > K. \end{cases}$$

Interpretation: the holder benefits when the asset price finishes above the strike.

European Put

The payoff of a European put option at maturity is

$$(K - S_T)^+ = \max(K - S_T, 0).$$

Equivalently,

$$(K - S_T)^+ = \begin{cases} K - S_T, & S_T < K, \\ 0, & S_T \geq K. \end{cases}$$

Interpretation: the holder benefits when the asset price finishes below the strike.

Call Options with Different Strikes I

Assume two European call options with the same maturity T and strikes

$$K_1 < K_2.$$

No-Arbitrage Relation

The call with the lower strike must be at least as expensive as the call with the higher strike:

$$C(K_1) \geq C(K_2).$$

Why? Suppose, on the contrary, that

$$C(K_1) < C(K_2).$$

Then we construct the following portfolio at time 0:

- buy one call with strike K_1,

Call Options with Different Strikes II

- sell one call with strike K_2,
- invest the initial cash inflow $C(K_2) - C(K_1)$ at the risk-free rate until time T.

The profit function of the portfolio is

$$V_T(S_T) = (S_T - K_1)^+ - (S_T - K_2)^+ + \big(C(K_2) - C(K_1)\big)e^{rT}.$$

Since $K_1 < K_2$, we always have

$$(S_T - K_1)^+ - (S_T - K_2)^+ \geq 0,$$

and also

$$\big(C(K_2) - C(K_1)\big)e^{rT} > 0.$$

Therefore,

$$V_T(S_T) > 0$$

Call Options with Different Strikes III

for every possible value of S_T.

Conclusion

This is an arbitrage opportunity. Hence, in an arbitrage-free market,

$$C(K_1) \geq C(K_2).$$

Put–Call Parity via Arbitrage I

Let C and P be the prices of a European call and a European put with the same strike K and maturity T, written on the same underlying asset. Assume a constant risk-free rate r and no dividends.

Put–Call Parity

In an arbitrage-free market,

$$C + Ke^{-rT} = P + S_0.$$

Case 1: Suppose that

$$C + Ke^{-rT} < P + S_0.$$

Construct the portfolio:

- buy one call,
- invest Ke^{-rT} in the bank account,
- sell one put,

Put–Call Parity via Arbitrage II

- sell one share of the underlying asset.

Its initial cost is

$$V_0 = C + Ke^{-rT} - P - S_0 < 0.$$

Hence the initial cash inflow is

$$-(V_0) = P + S_0 - C - Ke^{-rT} > 0.$$

Invest this amount at the risk-free rate until time T.
Let $x = S_T$. The terminal profit function is then

$$\Pi(x) = (x - K)^+ + K - (K - x)^+ - x + \left(P + S_0 - C - Ke^{-rT}\right)e^{rT}.$$

Since

$$(x - K)^+ - (K - x)^+ = x - K,$$

Put–Call Parity via Arbitrage III

we obtain

$$\Pi(x) = \left(P + S_0 - C - Ke^{-rT}\right)e^{rT} > 0 \qquad \text{for every } x > 0.$$

Therefore,

$$\Pi(x) \geq 0 \quad \text{for all } x > 0,$$

and in fact

$$\Pi(x) > 0 \quad \text{for all } x > 0.$$

So we have an arbitrage opportunity.

Case 2: Suppose that

$$C + Ke^{-rT} > P + S_0.$$

Construct the reverse portfolio:

- sell one call,
- borrow Ke^{-rT},

Put–Call Parity via Arbitrage IV

- buy one put,
- buy one share of the underlying asset.

Its initial cost is

$$V_0 = -C - Ke^{-rT} + P + S_0 < 0,$$

so again we receive a strictly positive amount at time 0, namely

$$C + Ke^{-rT} - P - S_0 > 0.$$

Investing this amount until time T, the terminal profit becomes

$$\Pi(x) = -(x-K)^+ - K + (K-x)^+ + x + \left(C + Ke^{-rT} - P - S_0\right)e^{rT}.$$

Using again

$$(x-K)^+ - (K-x)^+ = x - K,$$

Put–Call Parity via Arbitrage V

we get

$$\Pi(x) = \left(C + Ke^{-rT} - P - S_0\right)e^{rT} > 0 \qquad \text{for every } x > 0.$$

Hence,

$$\Pi(x) \geq 0 \quad \text{for all } x > 0,$$

and

$$\Pi(x) > 0 \quad \text{for all } x > 0.$$

So this also yields an arbitrage opportunity.

Conclusion

Both strict inequalities lead to arbitrage. Therefore, the only possible relation in an arbitrage-free market is

$$C + Ke^{-rT} = P + S_0.$$

Definition of Arbitrage I

Deterministic Arbitrage

An arbitrage opportunity is a trading strategy whose terminal profit function is non-negative in every possible state and strictly positive in at least one state.

Assume that the initial cost of constructing the portfolio is Y, and let $G(x)$ denote its terminal payoff when $x = S_T$.
We define the terminal profit function by

$$\Pi(x) = G(x) - Ye^{rT}.$$

Here, the term Ye^{rT} represents the time-T value of the initial cost. Thus, $\Pi(x)$ includes both:

- the payoff generated by the portfolio at maturity, and
- the financing effect of the initial cost.

Definition of Arbitrage II

Therefore, an arbitrage opportunity means

$$\Pi(x) \geq 0 \quad \text{for every } x \geq 0,$$

and

$$\Pi(x) > 0 \quad \text{for some } x \geq 0.$$

Interpretation

If $Y < 0$, the investor receives money at time 0 and may invest it at the risk-free rate until time T. If $Y > 0$, the investor must finance this cost. The function $\Pi(x)$ measures the final profit after taking this initial cost into account.

Linear Programming Detection of Arbitrage I

Assume that the strikes are ordered as

$$0 < K_1 < \cdots < K_n.$$

Consider portfolios formed from

stock,   bank account,   European calls and puts.

If $x = S_T$, the terminal profit function is

$$\Pi(x) = ax + be^{rT} + \sum_{i=1}^{n} \gamma_i (x - K_i)^+ + \sum_{i=1}^{n} \delta_i (K_i - x)^+ - Ye^{rT},$$

where Y is the initial cost of constructing the portfolio.

Linear Programming Detection of Arbitrage II

Linear Program

We solve

$$\min D$$

subject to

$$aS_0 + b + \sum_{i=1}^{n} (\gamma_i C(K_i) + \delta_i P(K_i)) = Y,$$

$$\Pi(x) \geq -D \qquad \text{for all } x \geq 0,$$

$$a, \gamma_i, \delta_i \in [-N_i, N_i].$$

This linear program always has a feasible solution, because investing the amount Y in the bank account yields $D = 0$. Therefore, the program identifies the portfolio that achieves the minimum value of D.

Linear Programming Detection of Arbitrage III

Why does $\Pi(x) \geq -D$ give linear constraints?

For each fixed x, the quantities

$$x, \qquad (x - K_i)^+, \qquad (K_i - x)^+$$

are constants. Hence the inequality

$$\Pi(x) \geq -D$$

is linear in the decision variables

$$a,\ b,\ \gamma_i,\ \delta_i,\ Y,\ D.$$

Linear Programming Detection of Arbitrage IV

Why only finitely many constraints?

The function $\Pi(x)$ is piecewise linear, with breakpoints

$$0,\ K_1,\ \ldots,\ K_n.$$

Hence:

- on each bounded interval, it is enough to check the endpoints;
- on $[K_n, \infty)$, we have

$$\Pi(x) = mx + c, \qquad m = a + \sum_{i=1}^{n} \gamma_i - \sum_{i=1}^{n} \delta_i,$$

  so it is enough to require

$$\Pi(K_n) \geq -D \qquad \text{and} \qquad m \geq 0.$$

This linear program detects a stronger notion of arbitrage than the one in the definition. If

$$D^* < 0,$$

then

$$\Pi(x) \geq -D^* > 0 \qquad \text{for all } x \geq 0.$$

Hence the profit is strictly positive in every state, not merely in some state.

Why Deterministic Arbitrage Matters for Pricing I

Main Point

Deterministic arbitrage is the notion of arbitrage that actually shapes prices. It imposes hard, model-free restrictions on the market, because a riskless profit opportunity can be scaled and replicated, creating immediate trading pressure until the mispricing disappears.

This is why relations such as

put–call parity, monotonicity in the strike, and convexity across strikes

are not merely technical formulas: they are consequences of the absence of deterministic arbitrage.

Why Deterministic Arbitrage Matters for Pricing II

By Contrast

Statistical arbitrage is different. It refers to trading strategies that may be profitable in expectation, or with high probability, under a statistical model. But such strategies may also generate losses over finite horizons.

Therefore, statistical arbitrage does not determine prices in the same sense. It does not produce hard pricing identities or model-free bounds. Instead, it is simply one possible trading methodology among infinitely many others, whose success depends on assumptions, estimation, and market conditions.

Conclusion

Deterministic arbitrage is the grammar of pricing. Statistical arbitrage is a speculative trading approach, not a pricing principle.

Arbitrage Optimized on a Predicted Region I

Assume that the investor expects

$$S_T \in A \subseteq \mathbb{R}_+, \qquad A \neq \varnothing.$$

Instead of merely detecting arbitrage, one may search for an arbitrage portfolio that is as profitable as possible on A.

Arbitrage Optimized on a Predicted Region II

Linear Program

We solve

$$\max t$$

subject to

$$aS_0 + b + \sum_{i=1}^{n} (\gamma_i C(K_i) + \delta_i P(K_i)) = Y,$$

$$\Pi(x) \geq 0 \qquad \text{for all } x \geq 0,$$

$$\Pi(x) \geq t \qquad \text{for all } x \in A,$$

$$a, \gamma_i, \delta_i \in [-N_i, N_i].$$

Interpretation

Arbitrage Optimized on a Predicted Region III

If $t^* > 0$, then

$$\Pi(x) \geq 0 \quad \text{for all } x \geq 0,$$

and

$$\Pi(x) > 0 \quad \text{for all } x \in A.$$

Hence the portfolio is an arbitrage opportunity, and moreover it is uniformly profitable on the predicted region A.

See [1] for the derivation of linear constraints corresponding to the requirement $\Pi(x) > 0$ for all $x \in A$.

References

[1] N. Halidias,
Financial Engineering with Python,
forthcoming, World Scientific Publishing.
Draft available on ResearchGate:
`https://www.researchgate.net/publication/394413705_Financial_Engineering_with_Python`.

[2] N. Halidias,
Financial-Engineering,
GitHub repository with Python code accompanying *Financial Engineering with Python*.
Available at:
`https://github.com/nikoshalidias/Financial-Engineering`.
Includes, among others, the notebooks `PCUP.ipynb` (one-asset portfolio construction) and `Multi_Asset4.ipynb` (two-asset construction with bid–ask spread).