Κεφάλαιο 6ο: Όρια ακολουθιών και συναρτήσεων. Σειρές Taylor.

H entolή me thn opoίa brίskoume to όrio miaV akolouqίaV eίnai h Limit.

```
?Limit
```

```
Limit@expr, x->x0D finds the
   limiting value of expr when x approaches x0. More…
```

Me ??Limit **brίskoume ta carakthristikά thV entolήV** Limit

```
?? Limit
```

```
Limit@expr, x->x0D finds the
   limiting value of expr when x approaches x0. More…
```

```
Attributes@LimitD = 8Listable, Protected<

Options@LimitD = 8Analytic → False, Direction → Automatic<
```

To protected **shmaίnei όti den mporoύme na thn allάxoume enώ to** Listable **shmaίnei όti mporeί na ejarmosteί sugcrόnwV se pollέV akolouqίeV dhl. se mia olόklhrh lίsta apo akolouqίeV. P.c**

$$\text{LimitA9}\frac{\text{Log@nD}}{\text{n}},\ \text{è!!!}\ \text{n},\ \frac{1-\text{n}^3}{\text{n}^2}=,\ \text{n}\to\infty\text{E}$$

```
80, 1, -∞<
```

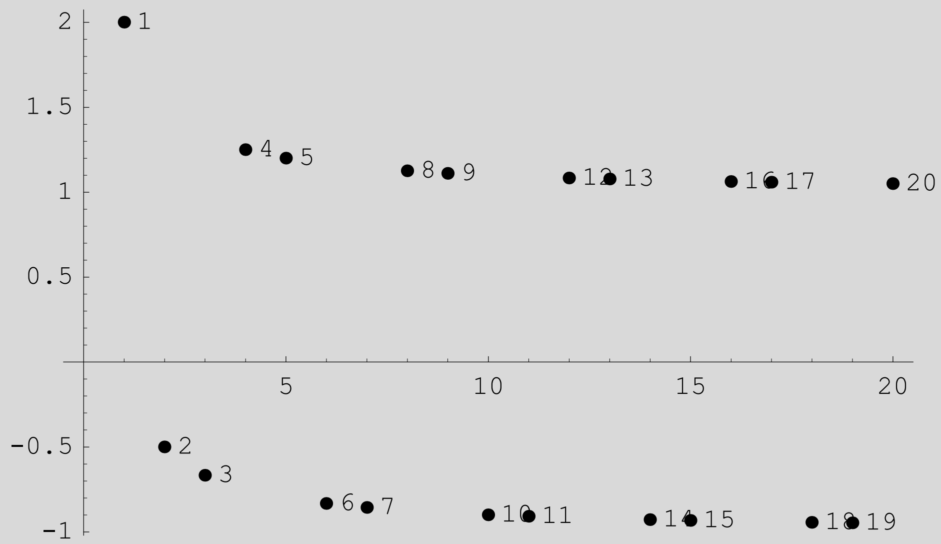
Όpwv **blέpoume h** [illegible] **eίnai apoklίnousa. Eάn h akolouqίa έcei dύo ή perissόtereV sugklίnouseV upoakolouqίeV tόte apantάei me έna mήnuma thV morjήV** Interval[{a,b}] **pou shmaίnei όti upάrcoun kάpoieV piqanέV lύseiV sto diάsthma** [**a**,**b**]. **p.c**

$$\text{LimitA}\frac{1}{\text{n}}+\text{SinA}\frac{\text{n}\,\pi}{2}\text{E}+\text{CosA}\frac{\text{n}\,\pi}{2}\text{E},\ \text{n}\to\infty\text{E}$$

```
Interval@8-2, 2<D
```

H parapάnw akolouqίa έcei duo oriakoύV ariqmoύV to 1 **kai to** -1. ΆV **kάnoume kai έna diάgramma na to doύme**:

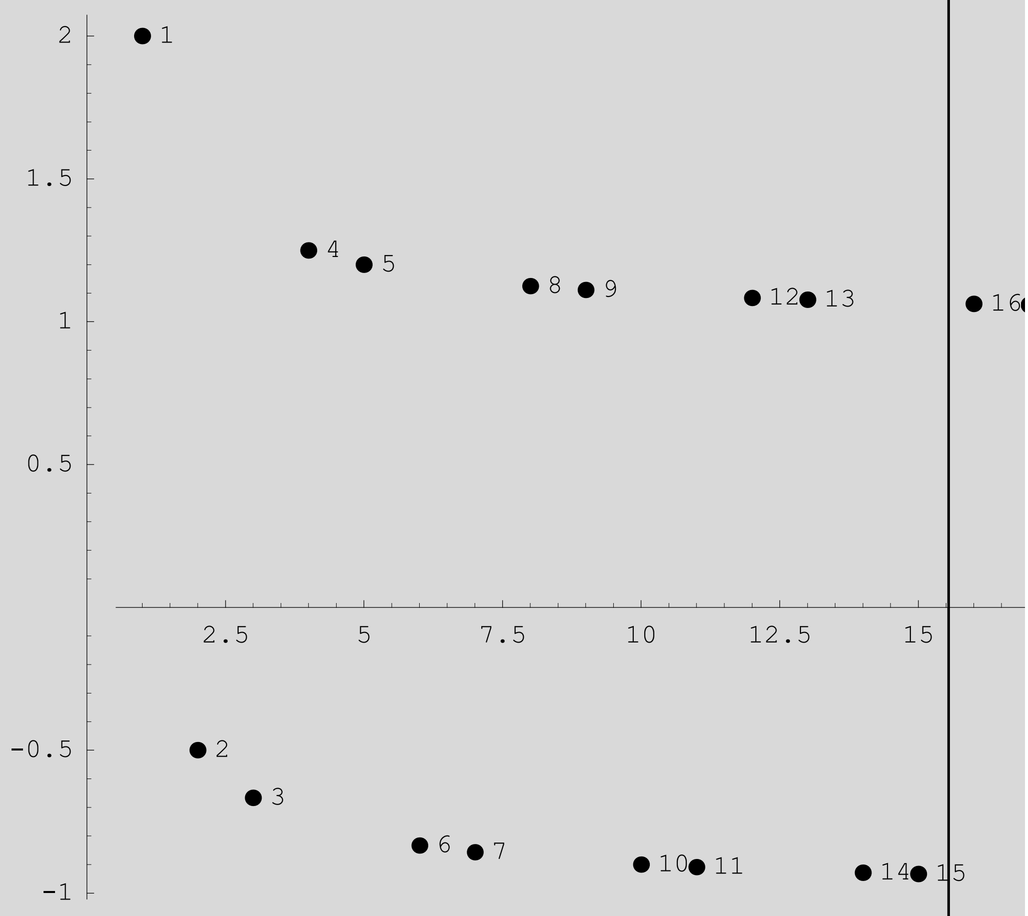
```
Remove@LabeledListPlotD

                              1          n π           n π
oiPrwtoi20oroi = TableA ¡¡¡¡ + SinA ¡¡¡¡¡¡ E + CosA ¡¡¡¡¡¡ E, 8n, 20<E;
                              n           2             2
<< Graphics`Graphics`
LabeledListPlot@oiPrwtoi20oroiD
```



¦ Graphics ¦

Epeidή **den apeikon**ί**zontai** ό**la ta shme**ί**a swst**ά, **all**ά**zoume ta carakthristik**ά **thV** LabeledListPlot:

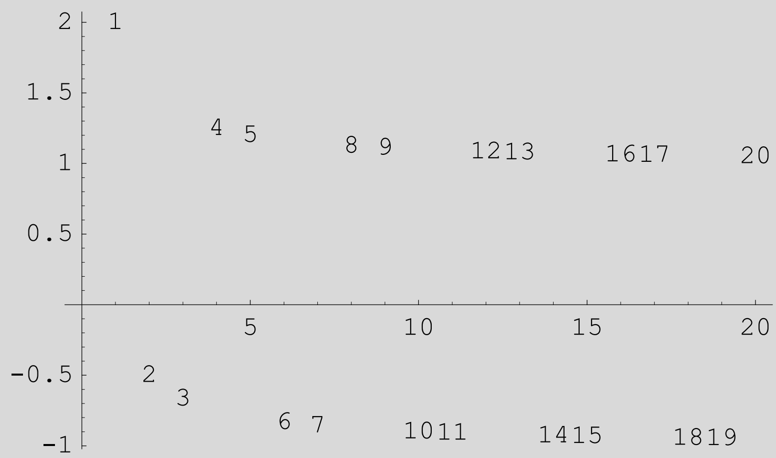
```
Remove@LabeledListPlotD

                              1            n π         n π
oiPrwtoi20oroi = TableA ¡¡¡¡ + SinA ¡¡¡¡¡¡¡ E + CosA ¡¡¡¡¡¡¡ E, 8n, 20<E;
                              n             2           2
<< Graphics`Graphics`
LabeledListPlot@oiPrwtoi20oroi,
 PlotRange → All, AxesOrigin → 80, 0<, AspectRatio → 0.8D
```



¦ Graphics ¦

Upárcei kai állo V énaV trópoV na doúme ta parapánw shmeía me thn crήsh thV TextListPlot

```
TextListPlot[Table[1/n + Sin[n π/2] + Cos[n π/2], {n, 20}]]
```




```
- Graphics -
```

Upάrcei kai mia ceirόterh perίptwsh pou enώ upάrcei to όrio toMathematica adunateί na apantήsei ή na dώsei lάqoV apάnthsh p.c

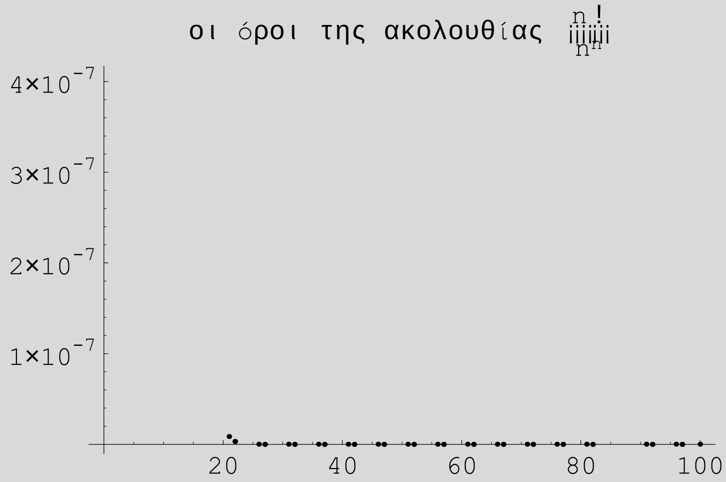
```
Limit[{n!/n^n, √(n!)}, n → ∞]
```

```
Series::esss : Essential singularity encountered in Gamma[1/n + 1 + O[n]^3].

Series::esss : Essential singularity encountered in Gamma[1/n + 1 + O[n]^3].

Series::esss : Essential singularity encountered in Gamma[1/n + 1 + O[n]^3].

General::stop :
 Further output of Series::esss will be suppressed during this calculation.
```

```
{Limit[n^-n n!, n → ∞], Limit[√(n!), n → ∞]}
```

Eίnai gnwstό όti h prώth eίnai mhdenikή kai h deύterh apoklίnousa sto +¶.

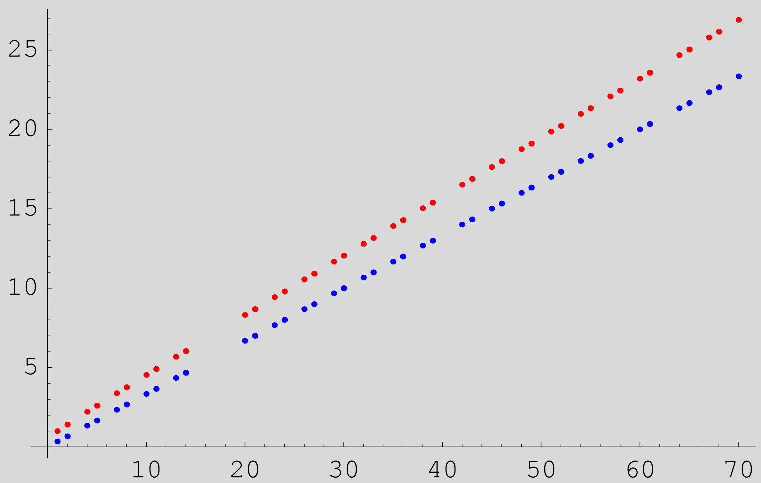
```
ListPlotATable@8n, n^-n n!<, 8n, 100<D,
 PlotLabel -> "οι όροι της ακολουθίας n!/n^n"E
```




```
¦ Graphics ¦
```

AV scediάsoume tώra thn $n!^{\frac{1}{n}}$. **H** $n!^{\frac{1}{n}}$ **sumperijέretai grammikά me klίsh "perίpou" eίnai ίsh me** 1/3. **Gia na to apodeίxoume autό**(**grajikή apόdeixh**!) **scediάzoume kai thn** $\frac{n}{3}$ **me diajoretikά crώmata gia na doύme thn diajorά** :

```
timesToyn = 70
Remove@DisplayTogetherD
<< Graphics`Graphics`
p3 =
 DisplayTogetherAListPlotATableA9n, n!^(1/n)=, 8n, 1, timesToyn<E,
   PlotStyle → RGBColor@1, 0, 0DE, ListPlotA
   TableA9n, n/3 =, 8n, 1, times<E, PlotStyle → RGBColor@0, 0, 1DEE

70
```




```
⁞ Graphics ⁞
```

Genikά **h cr**ή**sh twn grajik**ώ**n parast**ά**sewn e**ί**nai** έ**na cr**ή**simo ergale**ί**o.** Ό**pou e**ί**nai ejikt**ό **qa d**ί**noume kai mia katt**ά**llhlh grajik**ή **par**ά**stash.**

6.1 Aqroísmata, Ginómena,SeirέV

Tό ά**qroisma d**ί**netai apo thn sun**ά**rthsh** Sum. **Ta** ό**ria tou aqro**ί**smatoV d**ί**nontai me k**ά**poieV tim**έ**V sta** min **kai** max. **To** d , **an up**ά**rcei parist**ά**nei to b**ή**ma pou aux**ά**nontai oi tim**έ**V thV metablht**ή**V** i. **Sunoy**ί**zontaV** έ**coume tiV parak**ά**tw morj**έ**V**

Sum[f,{i,max}] ή Sum[f,{i,min,max}] ή Sum[f,{i,min,max,d}]

Paradeί**gmata**:

```
seira = SumA x^i/i , 8i, 1, 10, 2<E

x + x^3/3 + x^5/5 + x^7/7 + x^9/9
```

```
seira^2
```

$$\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\frac{x^9}{9}\right)^2$$

```
Expand@seira^2D
```

$$x^2+\frac{2x^4}{3}+\frac{23x^6}{45}+\frac{44x^8}{105}+\frac{563x^{10}}{1575}+\frac{124x^{12}}{945}+\frac{143x^{14}}{2205}+\frac{2x^{16}}{63}+\frac{x^{18}}{81}$$

ΌpwV blέpoume mia seirά mporoύme thn ceiristoύme όpwV qέloume. P.c na broύme to tetrάgwnό thV kai epίshV mporoύme kai na thn paragwgήsoume ή na oloklhrώsoume!

```
D@seira, 8x, 2<DH  deyterh paragwgos ws pros x  L
integ = Integrate@seira, xDH  aoristo oloklhrwma ws pros x L
```

$$2x+4x^3+6x^5+8x^7$$

$$\frac{x^2}{2}+\frac{x^4}{12}+\frac{x^6}{30}+\frac{x^8}{56}+\frac{x^{10}}{90}$$

H sunάrthsh Coefficient eίnai polύ crήsimh diόti όtan to anάptugma thV seirάV eίnai arketά megάlo mporeί na maV breί ton suntelestή miaV dύnamhV x^n miaV metablhtήV x . P.c

```
Coefficient@integ, x, 10D
```

$$\frac{1}{90}$$

ΆlleV paralagέV tou Coefficient mporeίte na breίte patώntaV F1 kata ta gnwstά. Parakάtw paraqέtontai ta merikά aqroίsmata dunάmewn tou i se morjή pίnaka kai h gnwstή maV gewmetrikή prόodo me lόgo w.

```
TableA9m, Σ(i=1..n) i^m=, 8m, 1, 3<E êê TableForm
```

1	$\frac{1}{2}n(1+n)$
2	$\frac{1}{6}n(1+n)(1+2n)$
3	$\frac{1}{4}n^2(1+n)^2$

```
Sum@a ωi−1, 8i, 1, n<D

 a H−1 + ωn L
¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡
   −1 + ω
```

Gia na broúme an h parapánw gewmetrikή seirá sugklínei qétoume n **= ¶**

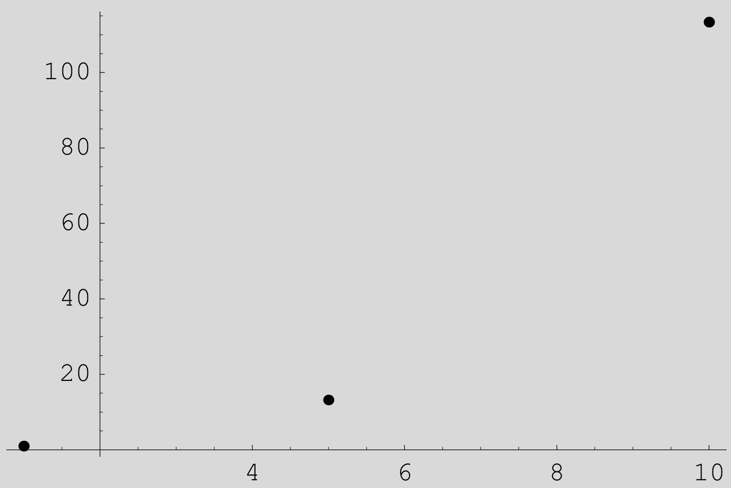
```
Sum@a ωi−1, 8i, 1, ∞<D

−   a
  ¡¡¡¡¡¡¡¡¡¡¡¡
  −1 + ω
```

Prέpei na prosέxoume ó**ti siwphlá h** Sum **upoqétei óti o paranomastήV** -1+**w έcei thn idióthta** |**w**|<1 **dióti alliώV den suglίnei**! **AV doúme ti gínetai se antíqeth períptwsh**

```
Sum@H1.5Li−1, 8i, 1, ∞<D

−2.
```

pou jusiká eínai láqoV(dióti ejarmózetai láqoV o túpoV $-\frac{1}{1+\omega}$ **gia w**=1.5). **AV doúme lígo kai thn grajikή parástash twn merikώn aqroismátwn gia na to epibebaiώsoume**

```
ListPlot@Table@8n, Sum@H1.5Li−1, 8i, 1, n<D<, 8n, 10<D,
 PlotStyle → AbsolutePointSize@4DD
```




```
¦ Graphics ¦
```

To súmbolo tou apeírou mporeí na eisacqeí grájontaV \[Infinity] (**metaxú tou** Infinity **kai**] **den prέpei na upárcei kenó gιatí den qa metarapeí autómata se ¶**) ή **patώntaV to plήktro** Esc **kai metá**

grάjontaV inf kai xanapatώntaV to Esc crhsimopoiώntaV thn basikή palέta 3 BasicInput. Όmoia to /$_{n=0}^{m}$ ¡¡¡$\frac{1}{f@nD}$¡¡¡ mporeί na grajeί patώntaV ta plήktra Â sum Â , Î+Ï n=0 , Î%Ï m , Îâ Ï 1 , /f[n] , Îâ Ï (ta [kai] den ta ktupάme! to , Î+]gia parάdeigma shmaίnei όti patάme mazί to , kai to + enώ to , Îâ Ï eίnai to , mazί me to SPACE plήktro) . To Limit adunateί na breί όmwV thn άpeirh seirά όtan upάrcei kάpoia staqerά:

```
Limit@Sum@aω^(i-1), 8i, 1, n<D, n → ∞D

LimitA (-1 + aω^n)/(-1 + aω), n → ∞E
```

Gia na broύme to άqroisma thV seirάV me Limit qa prέpei angkastikά na bάloume kάpoieV timέV sta a kai w p.c

```
LimitA (-1 + 2 H-1/3L^n)/(-1 + 2 H-1/3L), n → ∞E

5/7
```

Me Sum[f,{i,n,m},{j,k,l}] mporoύme na pάroume aqroίsmata thV f [i,j] όtan έcoume pάnw apo mia metablhtή(edώ έcoume thn i kai thn j) p.c.

```
Sum@x^i y^j, 8i, 1, 4<, 8j, 1, 2<D

x y + x^2 y + x^3 y + x^4 y + x y^2 + x^2 y^2 + x^3 y^2 + x^4 y^2
```

H basikή entolή pou upologίzoume ginόmena eίnai h

Product[f,{i,imin,imax}]

Mporoύme na broύme kai diplά ginόmena ή ta όria gia imax->¶ όpwV gίnetai kai me to Sum. Paradeίgmata:

```
ProductA Hi + 1L^2/(i Hi + kL), 8i, 1, n<E

ProductA Hi + 1L^2/(i Hi + kL), 8i, 1, ∞<E

H1 + nL Gamma@1 + kD Gamma@2 + nD / Gamma@1 + k + nD
```

```
Gamma@1 + kD
```

Άskhsh: **H sunάrthsh j**(x) **tou** Euler **dίnei to plήqoV twn akeraίwn** n **metaxύ tou** 1 **kai tou** x **oi opoίoi den έcoun kanένa koinό diairέth me ton** x. **Sto** *Mathematica* **h sunάrthsh autή dίnetai me thn** Euler-Phi[x] **p.c**

```
EulerPhi@60D

16
```

Sth Qewrίa Ariqmώn dίnetai me ton tύpo $j\text{H}x\text{L} = x$ ‰ $\text{I}1 - \frac{1}{p}\text{M}$ **ή me άlla lόgia eίnai to** x **epί to ginόmeno** (pêx) **twn όrwn** $1 - \frac{1}{p}$ **όpou to** p **eίnai έnaV prwtoV diairέthV tou** x **mikrόteroV jusikά apo to** x. **Na orίsetai mia sunάrthsh**
phiEuler[x_Integer] **pou qa eίnai akribώV to parapάnw ginόmeno Elέgxte ta apotelέsmata saV me thn boήqeia thV** EulerPhi.

6.2 Όria sunarthσewn miaV metablhtήV

Έstw f(x) **mia sunάrthsh kai qέloume na upologίsoume to όrio thV kaqώV to** x **teίnei sto** x_0. **Autό grάjetai sto** *Mathematica* Limit[f@xD, x **->** x_0]. **P.c**

```
Limit[ (9 x^2 - 1) / (9 x^2 + 51 x - 18) , x → 1/3 ]

2/19
```

Όtan to όrio den upάrcei ή adunateί na to breί, tόte bgάzei to mήnuma Interval[{a,b}] ή **kάti άllop.c**

```
Limit[Sin[1/x], x → 0]

Interval@8-1, 1<D
```

To Interval[{-1,1}] **maV bebaiώnei όti sίgoura den upάrcei to όrio kai όti ίswV upάrcoun όria upoakolouqiώn metaxύ tou** -1 **kai** 1. **Genikά όtan douleύomai me ta όria prέpei na eίmaste prosektikoί. Upάrcei dustucώV kai h perίptwsh pou to** *Mathematica* **dίnei lάqoV όrio. Gia parάdeigma aV pάroume thn** f@xD = $\frac{\text{Abs}[x-2]}{x-2}$ **pou jusikά paίrnei timέV** +1 **apo dexiά kai** -1 **apό aristerά. Opόte to όrio** Limit[f[x],x->2] **den qa έprepe na upάrcei. To** *Mathematica* **όmwV dίnei lάqoV apάnthsh**:

```
        Abs@x − 2D
f@xD := ¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡
          x − 2
Limit@f@xD, x → 2D

1
```

Gia touV parapánw lógouV qa prépei na paírnoume merikéV jorév kai ta pleuriká ória gia na diapistώnoume an parousiázetai kápoia asunéceia ή óci.To ório apo ta aristerá to paírnoume me Direction->1 enώ apo ta dexiá me Direction->-1.

```
Limit@f@xD, x → 2, Direction → 1D
Limit@f@xD, x → 2, Direction → −1D

−1
```

```
1
```

6.3 Diplá ória sunarthsewn dúo metablhtώn

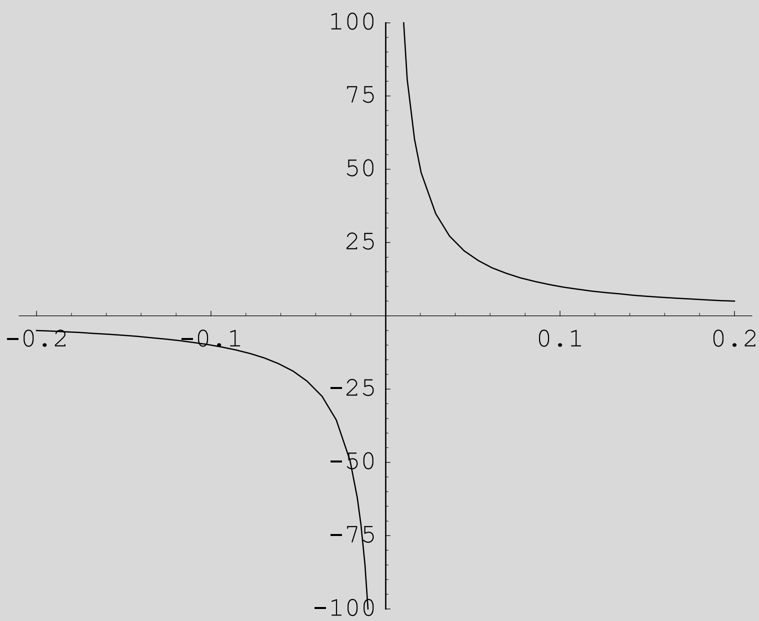
To Mathematica maV parécei thn dunatóthta eúreshV móno twn diplώn orίwn Limit[Limit[f[x,y],y->y0],x->x0] kai Limit[Limit[f[x,y],x->x0],y->y0]. Den maV parécetai h dunatóthta na broúme to ório Limit[f[x,y],(x,y)->(x0,y0)]. Autá ta ória eínai crήsima gia na bgáloume kápoia sumperásmata gia thn sumperijorá miaV sunárthshV kontá sto shmeío (x0,y0). Gia parádeigma an ta ória Limit[-Limit[f[x,y],y->y0],x->x0] kai Limit[Limit[f[x,y],x->x0],y->y0] ótan upárcoun allá den eínai ísa tóte den mporeí na upárcei tó dipló ório. Antístroja an upárcei to ório tóte ta diplá eínai ísa. AV pároume gia parádeigma thn f[x,y]= $\frac{x^2+y}{x+y}$ kai aV broúme ta diplá ória sto (0,0):

```
Remove@f, x, yD
              x^2 + y
f@x_, y_D := ¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡
             x^2 + y^2
Limit@Limit@f@x, yD, x → 0D, y → 0D
Limit@Limit@f@x, yD, y → 0D, x → 0D

∞
```

```
1
```

Ta pleuriká ória kai oi grajikéV parastáseiV mporoún na maV bohqήsoun na katanoήsoume kalútera thn sumperijorá thV f gúrw apo éna shmeío (x0,y0). Edώ dínontai kápoia paradeígmata me thn Plot. An duskoleúeste kai zhtáte boήqeia, mporeítai na metajéretai ton kérsora pánw se mia sunárthsh ή se kápoia epilogή thV(p.c pánw sthn Plot) kai metá patήste F1 gia na párete boήqeia kai paradeígmata scetiká me thn sunárthsh pou qélete boήqeia.

```
Remove@fD
                                      x^2 + y
f@x_, y_D := IfAx^2 + y^2 ≠ 0, ¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡ , 0E
                                     x^2 + y^2
H  αν ο παρανομαστής είναι 0 τότε θέτουμε τιμή =0 L
x = 0;
                          x^2 + y
ymin = −.2; ymax = .2; ¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡¡
                         x^2 + y^2
Plot@f@x, yD, 8y, ymin, ymax<, PlotRange → 8−100, 100<,
 PlotPoints → 50, AxesOrigin → 80, 0<,
 AspectRatio → .8, MaxBend → 20, Compiled → FalseD

1
¡¡¡
y
```



¦ Graphics ¦

Bάlame skόpima perissόtera carakthristikά (options) sthn Plot apo όsa prάgmati creiάzontai ώste na mporeί kάpoioV na mάqei plhrojorίeV gia autά mέsw tou plήktrou F1. Basikά creiάzetai h plhrojorίa PlotRangeØ{-100,100} giatί cwrίV autή ίswV den saV bgeί ena katanohtό grάjhma. Me thn PlotRange kόboume katά boύlhsh ton άxona twn x ή twn y. (edώ o 0y scediάzetai gia timέV -100 έwV 100 kai me AspectRatioØ.8 zhtάme to mήkoV tou kάqetou άxona na eίnai to 8/10 tou mήkouV tou orizόntiou). Apό to scήma parathroύme όti gia mikrέV timέV tou x kontά sto 0 p.c x=0 ta pleurikά όria kaqώV y->0 eίnai ¶ kai - ¶ antίstoica. AV to doύme thn apάnthsh me to Limit:

```
Limit@Limit@f@x, yD, x → 0D, y → 0, Direction → 1D
Limit@Limit@f@x, yD, x → 0D, y → 0, Direction → −1D
```

```
LimitALimitAIfAx^2 + y^2 ≠ 0, (x^2 + y)/(x^2 + y^2), 0E, y → 0E, x → 0, Direction → 1E
```

```
LimitALimitAIfAx^2 + y^2 ≠ 0, (x^2 + y)/(x^2 + y^2), 0E, x → 0E, y → 0, Direction → −1E
```

Den ta katάjere! Edώ den jtaίei to Limit allά o orismόV me to If pou dώsame gia thn f. To Limit den έcei prόblhma me mhdenikoύV paranomastέV h Plot όmwV mporeί na έcei lόgw tou trόpou pou scediάzei thn grajikή parάstash(paίrnei kάpoia shmeίa ston άxona Ox brίskei ta timέV thV f kai enwnei me euqύgramma tmήmata ta shmeίa pou prokύptoun!). AV antikatastήsoume loipόn thn f me to klάsma $\frac{x^2+y}{x^2+y^2}$ mέsa sto Limit:

```
LimitALimitA(x^2 + y)/(x^2 + y^2), x → 0E, y → 0, Direction → 1E
LimitALimitA(x^2 + y)/(x^2 + y^2), x → 0E, y → 0, Direction → −1E
```

```
−∞
```

```
∞
```

Sto *Mathematica* upάrcei kai h perίergh perίptwsh na upάrcei to όrio kai na "mhn upάrcoun" ta diplά όria. Ένa tέtoio kakό parάdeigma eίnai h sunάrthsh f[x,y]=x*Sin(1/y). To έna apo ta diplά όria eίnai to Interval[{0,0}](dhl. ousiastikά όrio to 0!!!):

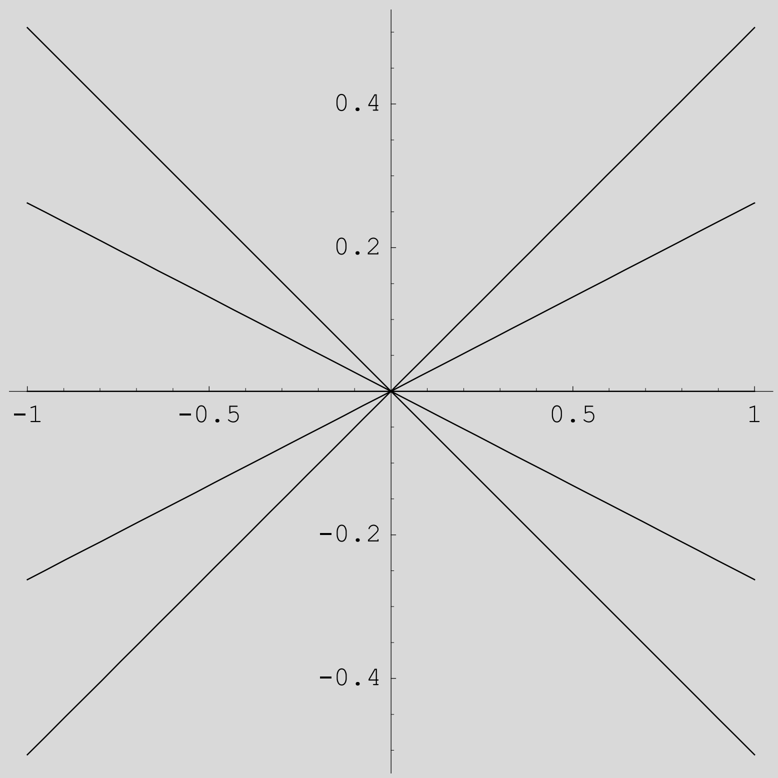
```
Limit@Limit@x Sin@1 ê yD, x → 0D, y → 0D
Limit@Limit@x Sin@1 ê yD, y → 0D, x → 0D
```

```
0
```

```
Interval@80, 0<D
```

AV kάnoume kai edώ thn grajikή parάstash gia arketέV timέV tou y kontά sto 0 gia na katalάboume thn apάnthsh Interval[{0,0}].

```
Remove@x, y, z, gD;
g@x_, y_D := If@y ≠ 0, x Sin@1 ê yD, 0D
xmin = -1; xmax = 1;
pinakas = Table@g@x, yD, 8y, -.02, .02, 1 ê 100<D
Plot@Evaluate@pinakasD, 8x, xmin, xmax<,
 PlotRange → All, AspectRatio → 1, PlotPoints → 40D

80.262375 x, 0.506366 x, 0, -0.506366 x, -0.262375 x<
```



¦ Graphics ¦

apo autά **bl**έ**poume** ό**ti oi tim**έ**V tou eswteriko**ύ **or**ί**ou** Limit[x Sin[1/y],yØ0] **den te**ί**noun se** έ**na** ό**rio** f[x] **dhlad**ή **se k**ά**poia sun**ά**rthsh tou** x (**bl**έ**pe kai** pinakas)**me apot**έ**lesma na e**ί**nai "ad**ύ**nato" na breqe**ί **kai to exwterik**ό ό**rio** Limit[Limit[x Sin[1/y],yØ0],xØ0]. Έ**tsi** ά**llote pa**ί**rnoume** ό**rio** 0 **apo dexi**ά H0+L **kai** ά**llote** 0 **apo aristerá**H0-L.**To** Evaluate **met**ά **thn** Plot **anagk**ά**zei na upologisto**ύ**n pr**ώ**ta** ό**leV oi sunart**ή**seiV tou** pinaka **prin ejarmoste**ί **h** Plot. **Diajoretik**ά **h** Plot **den qa mporo**ύ**se na k**ά**nei thn grajik**ή **par**ά**stash d**ί**oti qa n**ό**mize** ό**ti o** pinaka **e**ί**nai p**ί**nakaV kai** ό**ci k**ά**poieV sunart**ή**seiV**!)

Ά**skhsh: Ajo**ύ **bre**ί**te apo to** Help **tou** *Mathematica* **ti k**ά**noun oi parak**ά**tw sunart**ή**seiV na ermhne**ύ**setai me tiV grajik**έ**V parast**ά**seiV pou par**ά**gontai**, **thn sumperijor**ά **twn dipl**ώ**n or**ί**wn thV** g[x_,y_]:=If[y∫0,x Sin[1/y],0]. (**prosoc**ή **an pat**ή**sete dipl**ό **klik se k**ά**poia apo tiV grajik**έ**V parst**ά**seiV pou qa prok**ύ**youn qa de**ί**te** Movie!)

```
<< Graphics`Animation`
g@x_, y_D := If@y ≠ 0, x Sin@1 ê yD, 0D
MoviePlot@g@x, yD, 8y, −1, 1<,
 8x, −1, 1, 1 ê 20<, PlotRange → 8−1, 1<D
MoviePlot@g@x, yD, 8x, −1, 1<,
 8y, −1, 1, 1 ê 20<, PlotRange → 8−1, 1<D
```

6.4 Akrótata sunartήsewn

Gia na broúme **é**na topikó **el**á**cisto thV** f[x] **g**ú**rw apo to** x_0 **gr**á**joume** FindMinimum@f@xD, 8x, x_0}]. **Gia to topik**ó **m**é**gisto arke**í **na zht**ή**soume to el**á**cisto thV** -f[x] **di**ó**ti** maxf= -min[-f]. **An b**é**baia** é**coume sun**á**thsh peris**ó**terwn metablht**ώ**n t**ó**te g**í**nontai oi kat**á**llhleV allag**έ**V p.c gia na bro**ú**me to topik**ó **el**á**cisto g**ú**rw ap**ó **to shme**ío Hx_0, y_0L **gr**á**joume** FindMinimum@f@x, yD, 8x, x_0<, 8y, y_0}]. **AV do**ú**me tiV sunart**ή**seiV aut**έ**Vse sc**έ**sh me thn** g **pou** ή**dh melet**ή**same**:

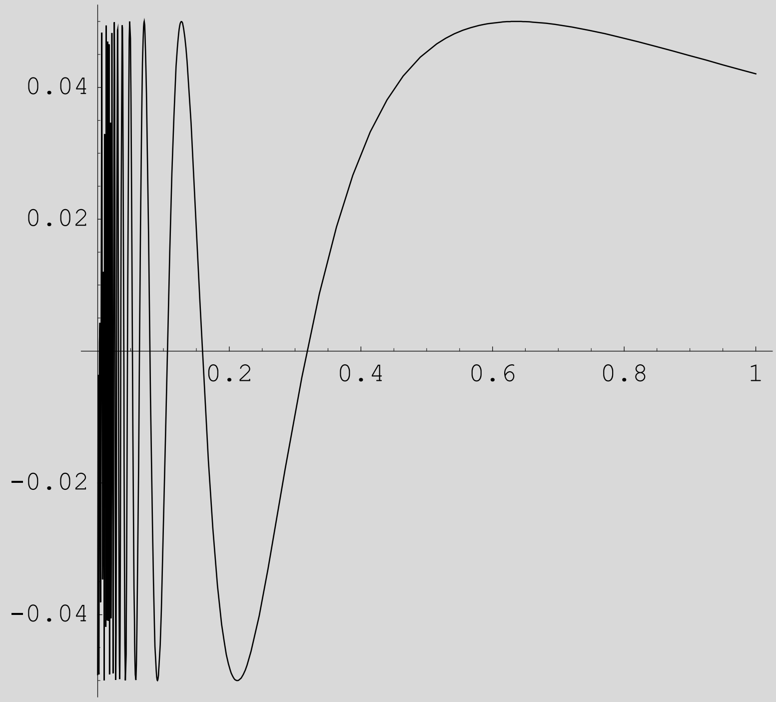
```
FindMinimum@g@.05, yD, 8y, 0.5<D
−FindMinimum@−g@.05, yD, 8y, 0.5<D
```

```
8−0.05, 8y → 0.212207<<
```

```
80.05, 8−Hy → 0.63662L<<
```

Dhladή **gia** x0=0.05 έ**coume m**έ**gisto** 0.05 **gia** y=0.63662 **kai el**ά**cisto**-0.05 **gia** y=0.212207.**AV k**ά**noume kai thn grajik**ή **par**ά**stash**:

```
ymin = 0; ymax = 1;
Plot@g@.05, yD, 8y, ymin, ymax<,
 PlotRange → All, AspectRatio → 1, PlotPoints → 40D
```




```
¦ Graphics ¦
```

An prospaqήsoume na broúme mέgisto kai elάcisto thV g **gia timέV tou** y **se έna diάsthma** [ymin,ymax] **qa prέpei na bάloume ta όria pou qa kiniέtai to** y **wV exhV**

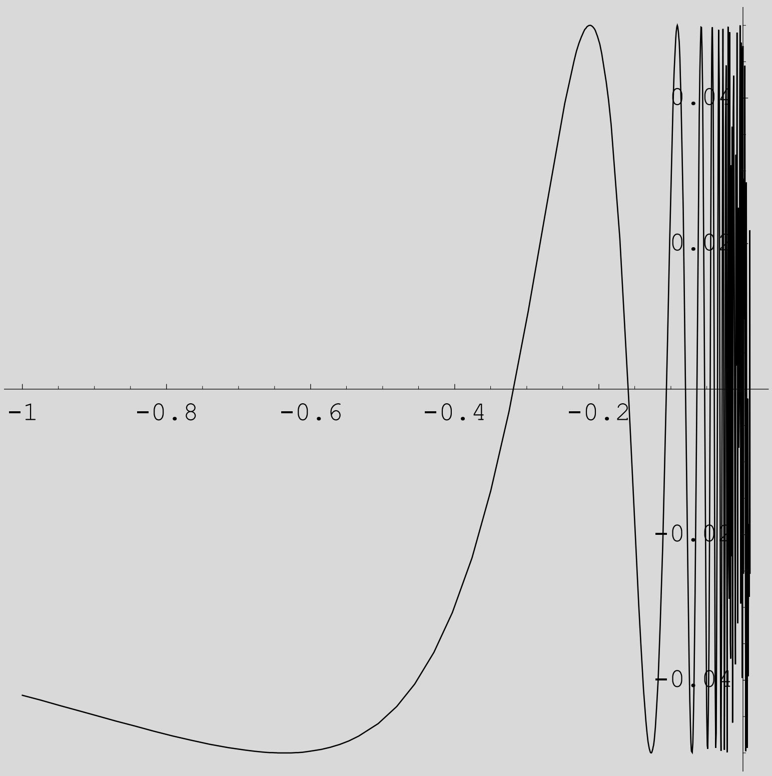
```
y0 = 0.005; ymin = −1; ymax = .01;
FindMinimum@g@0.05, yD, 8y, y0, ymin, ymax<D
−FindMinimum@−g@0.05, yD, 8y, y0, ymin, ymax<D
```

```
8−0.05, 8y → 0.00501275<<
```

```
80.05, 8−Hy → 0.00630317L<<
```

Apo autά blέpoume όti h eύresh tou topikoύ megίstou kai elacίstou exartάtai apo to shmeίo pou tou dίnomai. Έ**tsi gia** y_0 =0.5 έ**dwse** ά**lla akr**ό**tata kai gia** y_0 =0.005 ά**lla**.

```
Plot@g@0.05, yD, 8y, ymin, ymax<,
 PlotRange → All, AspectRatio → 1, PlotPoints → 40D
```




```
¦ Graphics ¦
```

An tώ**ra staqeropoi**ή**soume thn tim**ή **tou** y **p.c** y=.05 **bl**έ**poume** ό**ti e**ί**nai ad**ύ**naton na p**ά**roume k**ά**poia akr**ό**tata**:

```
x0 = 0.05; xmin = −1; xmax = 1;
FindMinimum@g@x, .05D, 8x, x0, xmin, xmax<D
```

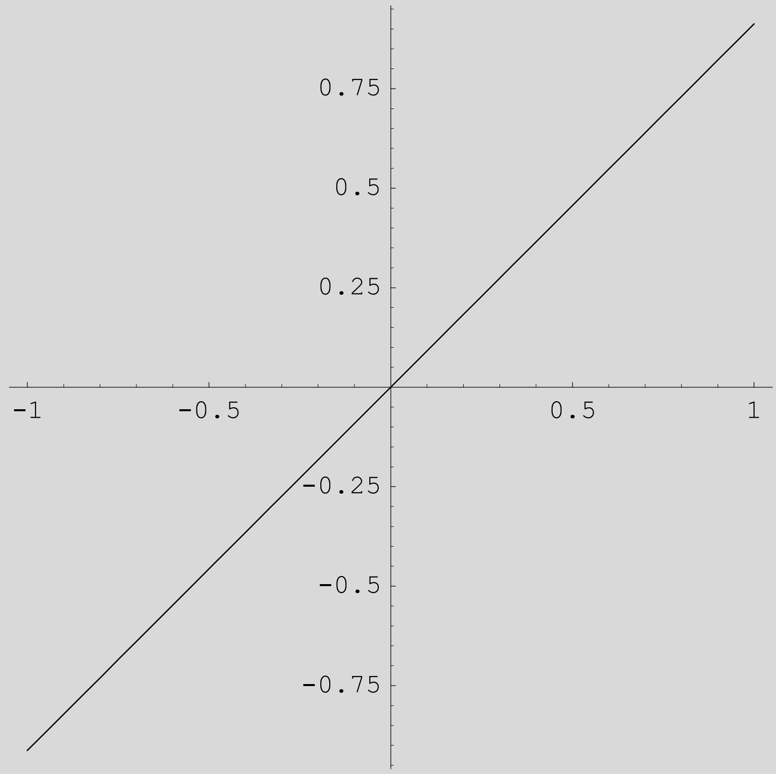
```
FindMinimum::regex :
 Reached the point 8−1.64894< which is outside the region 88−1., 1.<<.
```

```
FindMinimum@g@x, 0.05D, 8x, x0, xmin, xmax<D
```

Dhladή **den mp**ό**rese na bre**ί έ**na topik**ό **el**ά**cisto sta doq**έ**nta** ό**ria tou** x . **AV k**ά**noume p**ά**li thn grajik**ή **par**ά**stash gia na do**ύ**me ti akrib**ώ**V sumba**ί**nei**:

```
Remove@xD
g@x, 0.05D
Plot@g@x, 0.05D, 8x, xmin, xmax<,
 PlotRange → All, AspectRatio → 1, PlotPoints → 40D
```

```
0.912945 x
```




```
⁝ Graphics ⁝
```

prάgmati loipόn den έcei kάpoio akrόtato allά oi timέV g[x,0.05]megalώnoun ή mikraίnoun sunέceia! An pάroume kai ta akrόtata kai wV proV tiV dύo metablhtέV qa diapistώsoume ta ίdia:

```
x0 = 0.05; xmin = -1; xmax = 1; y0 = 0.005; ymin = -1; ymax = .01;
FindMinimum@g@x, yD, 8x, x0, xmin, xmax<, 8y, y0, ymin, ymax<D
```

```
FindMinimum::fmgs :
 Could not symbolically find the gradient of g@x, yD. Try using
   the default method, giving two starting values for each variable.
```

```
FindMinimum@g@x, yD, 8x, x0, xmin, xmax<, 8y, y0, ymin, ymax<D
```

6.5 SeirέV dunάmewn, SeirέV Taylor kai Mac-Laurin

Έstw f(x) sunάrthsh, pou έcei suneceίV paragώgouV wV proV x mέcri kai n tάxewV sto shmeίo a kai upάrcei h parάgwgoV n+1 tάxhV thV f sto a tόte upάrcei to anάptugma thV f se seirά gύrw apo to a se dunάmeiV H*x* - *a*L*ⁿ* p.c.gia n=4

```
Series@f@xD, 8x, a, 4<D
```

```
f@aD + fÅ@aD Hx − aL + 1/2 fÅÅ@aD Hx − aL^2 +
  1/6 f^H3L@aD Hx − aL^3 + 1/24 f^H4L@aD Hx − aL^4 + O@x − aD^5
```

To n+1=5 kai to O@x − aD^5 paristάnei to upόloipo(ή sjάlma) n+1 tάxhV thV f gia to shmeίo a kai έcei thn idiόthta to όrio kaqώV x->a na eίnai 0 dhl. Limit[O@x − aD^5,x->a]=0. O parapάnw tύpoV eίnai o tύpoV tou Taylor gia thn f. Eidikά όtan a=0 o tύpoV autόV gίnetai

```
Series@f@xD, 8x, 0, 4<D
```

```
f@0D + fÅ@0D x + 1/2 fÅÅ@0D x^2 + 1/6 f^H3L@0D x^3 + 1/24 f^H4L@0D x^4 + O@xD^5
```

pou den eίnai άlloV apo ton tύpo tou Mac Laurin gia thn f. Allά aV doύme kai paradeίgmata:

```
seira = Series@Log@xD, 8x, 1, 4<D
```

```
Hx − 1L − 1/2 Hx − 1L^2 + 1/3 Hx − 1L^3 − 1/4 Hx − 1L^4 + O@x − 1D^5
```

Thn parapάnw seirά mporoύme na thn uyώsoume sto tetrάgwno na thn paragwgίsoume wV proV x na thn oloklhrώsoume p.c

```
seira^2
D@seira, 8x, 2<DH η δευτερη παράγωγος ως προς x  L
Integrate@seira, xDH αόριστο ολοκληρωμα ως προς x  L
```

```
Hx − 1L^2 − Hx − 1L^3 + 11/12 Hx − 1L^4 − 5/6 Hx − 1L^5 + O@x − 1D^6
```

```
−1 + 2 Hx − 1L − 3 Hx − 1L^2 + O@x − 1D^3
```

```
1/2 Hx − 1L^2 − 1/6 Hx − 1L^3 + 1/12 Hx − 1L^4 − 1/20 Hx − 1L^5 + O@x − 1D^6
```

Χρήσιμες και στενά συνδεμένες με την Series είναι και οι Normal που αποκόπτει το υπόλοιπο O@x − aD^n απο το ανάπτυγμα Taylor και η SeriesCoefficient που επιστρέφει τον συντελεστή κάποιας δύναμης Hx − aL^m στο ανάπτυγμα Taylor. Ας δούμε τα παραδείγματα που ακολουθούν :

```
Normal@seiraD
SeriesCoefficient@seira^2, 4D
```

$-1-\frac{1}{2}\,H-1+xL^2+\frac{1}{3}\,H-1+xL^3-\frac{1}{4}\,H-1+xL^4+x$

$\frac{11}{12}$

TeleiώnontaV na poúme óti an h sunárthsh f **eínai duo metablhtώn tóte mporoúme na pároume thn diplή** Series. Έ**tsi h** Series[f, 8x, x_0, n_x<, 8y, y_0, n_y<] **brískei prώta to anáptugma wV proV to** y, **kai metá wV proV to** x.
P.c

```
Series@Sin@x yD, 8x, 0, 7<, 8y, 0, 7<D
```

$Hy+O@yD^8L\,x+\left(-\frac{y^3}{6}+O@yD^8\right)x^3+$
$J\frac{y^5}{120}+O@yD^8N\,x^5+J-\frac{y^7}{5040}+O@yD^8N\,x^7+O@xD^8$

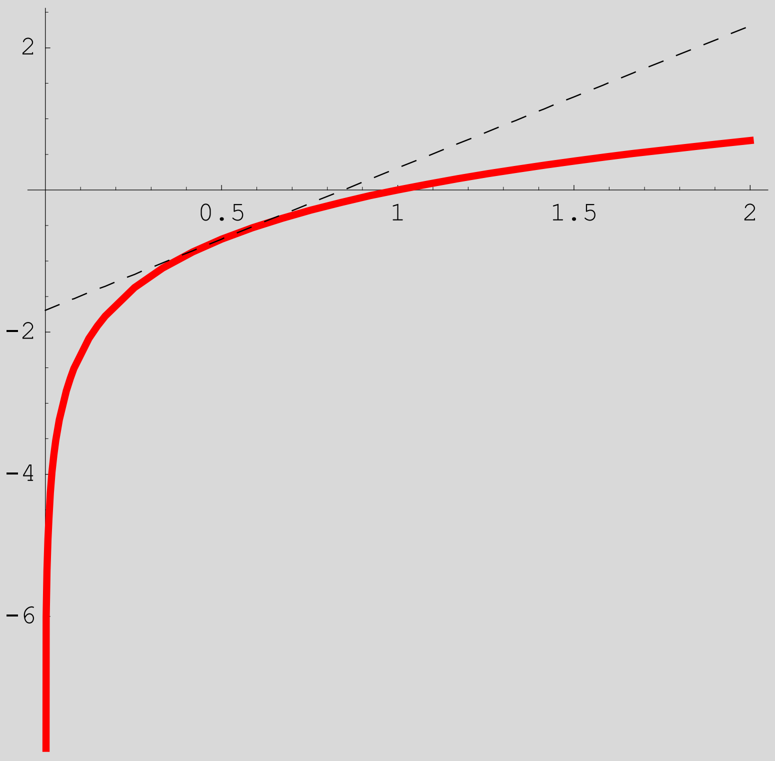
6.6 H ejaptómenh miaV sunárthshV

Me thn eukairía twn seirώn Taylor **thV** f **gúrw apo to shmeío** a **prépei na poúme óti ta poluwnumiká anaptúgmata pou prokúptoun me thn sunárthsh** Normal **proseggízoun polú kalá thn sunárthsh** f **gúrw apo to** a. ¨**Oso megalútero to** n **tóso kalúterh h proséggish**. **Ousiastiká loipón ta anaptúgmata** Taylor **apoteloún ta "ejaptómena" poluώnuma thV** f **sto** a. **Gia** n=1 **écoume thn ejaptómenh euqeía Parádeigma**: **gia na broúme thn ejáptomenh sthn** Log[x] **gia a**=0.5 **qétoume** n=1 **sthn** Series **kai metá thn paírnoume thn** Normal

```
efaptomenh = Normal@Series@Log@xD, 8x, .5, 1<DD
```

−0.693147 + 2. H−0.5 + xL

```
Plot@8Log@xD, efaptomenh<, 8x, 0, 2<,
 AxesOrigin → 80, 0<, AspectRatio → 1, PlotStyle →
  88RGBColor@1, 0, 0D, Thickness@.01D<, 8Dashing@80.02, 0.02<D<<D
```



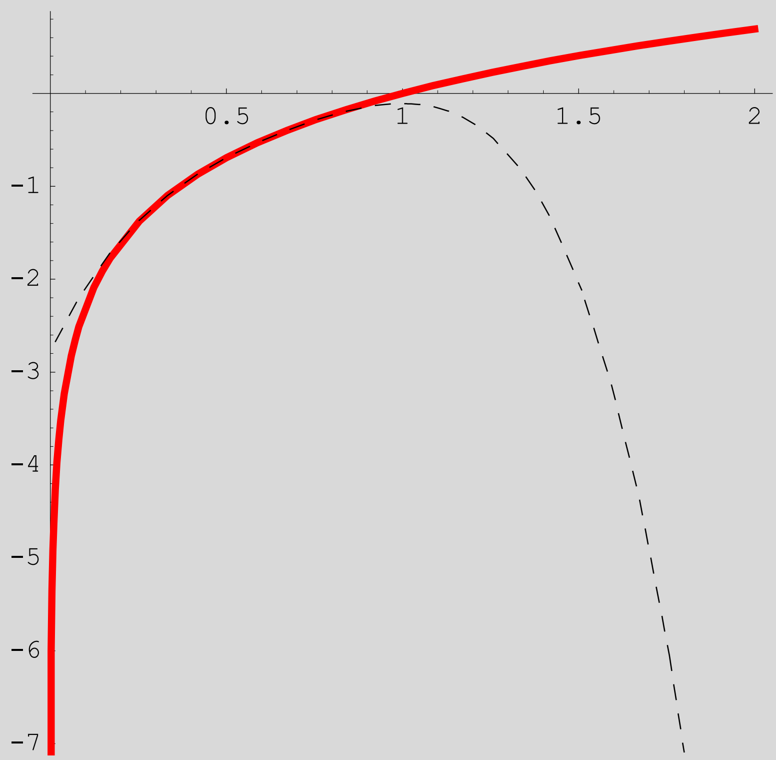

```
⁝ Graphics ⁝
```

Sto parapάnw grάjhma scediάsame mazί thn Log kai thn ejaptόmenh euqeία. To RGB[1,0,0] eίnai to kόkkino crwma kai to crhsimopoiήsame gia thn Log enώ to Dashing dhl. tiV diakekommέneV grammέV gia thn ejaptomέnh. Parathreίste όti prάgmati h euqeίa eίnai ejaptόmenh sto shmeίo a=0.5. Me crήsh thV Normal[Series[Log[x],{x,.5,n}]] gia n megalύtera tou 1 mporoύme na pάroume kalύtereV prosegistikέV kampύleV sthn Log. Mporeίtai na kάnetai thn grajikή parάstash gia n>1 gia na deίte thn diajorά. An tώra qέloume mia kalύterh prosέggish me poluώnumo p.c 4 baqmoύ (gύrw apo to shmeίo a jusikά)qa prέpei na broύme thn Series 4 baqmoύ dhladή:

```
efaptomenh4 = Normal@Series@Log@xD, 8x, .5, 4<DD
```

```
-0.693147 + 2. H-0.5 + xL -
 2. H-0.5 + xL^2 + 2.66667 H-0.5 + xL^3 - 4. H-0.5 + xL^4
```

```
Plot@8Log@xD, efaptomenh4<, 8x, 0, 2<,
 AxesOrigin → 80, 0<, AspectRatio → 1, PlotStyle →
  88RGBColor@1, 0, 0D, Thickness@.01D<, 8Dashing@80.02, 0.02<D<<D
```



¦ Graphics ¦

Άskhsh: **Dínetai h sunárthsh** f[x_]:= $\sqrt{2+x}$. **Breíte ta anaptúgmata** Mac-Laurin **thV** f[x] **baqmoú** n=1,2,...20 **me thn bohqeia thV** Series **kai omadopoiήste ta apotelésmata se mia lísta**. **Sthn sunéceia ejarmósete thn** Normal **sth lísta kai sthn lísta pou prokúptei qéste** x=0. **Qa prokúyoun** 20 **ariqmoí**. **Exhgeíste giatí autoí oi óroi apoteloún mia suglínousa akolouqía me ório to** $\sqrt{2}$. **Póso perípou eínai to** $\sqrt{2}$ **apo tiV proseggíseiV saV**;