BEHAVIOUR OF A NON-LOCAL EQUATION MODELLING LINEAR FRICTION WELDING

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ABSTRACT. A non-local parabolic equation modelling linear friction welding is studied. The equation applies on the half line and contains a nonlinearity of the form $f(u) \Big/ \left(\int_0^\infty f(u)dy\right)^{1+a}$. For $f(u) = e^u$, global existence and convergence to a steady state are proved. Numerical calculations are also carried out for this case and for $f(u) = (-u)^{1/a}$.

Key Words: Non-local parabolic problems.

2000 *Mathematics Subject Classification*: Primary 35B40, 35B45; Secondary 35Q72.

1. Introduction

In this paper we consider a non-local problem, on the half line, of the form

$$u_t = u_{xx} + f(u) \Bigg/ \left(\int_0^\infty f(u)\, dy\right)^{1+a} \qquad \text{for } 0 < x < \infty\,, \tag{1.1}$$

$$\text{with} \qquad u_x = 0 \text{ on } x = 0 \quad \text{and} \quad u_x \to -1 \text{ as } x \to \infty\,.$$

The problem arises as a local model for the temperature in a thin region which occurs during linear friction welding. An indication of how such a model arises is given in the following section, Sec. 2.

Similar equations applying in bounded domains have been previously considered as models for Joule heating and for thermo-viscous flow, [1], [2], [7], [8] or for chemotaxis, the aggregation of cells via interaction with a chemical substance, [14]. It has been seen that for $f(s)$ defined for all (positive) s and bounded away from zero, the solution for such a one-dimensional problem, subject, for example to homogeneous Dirichlet boundary conditions, exists globally and is bounded. These results can be extended to a two-dimensional radially symmetric problem and convergence proved for the special case of f being an exponential: $f(s) = e^s$.

A one-dimensional, finite-interval problem with f growing as a negative power near some value, *e.g.* $f(s) = (1-s)^{-p}$ for some positive p was considered by [11] Once again, global existence was proved.

In Sec. 3, we show that, subject to certain growth conditions at infinity, with f an exponential, the problem on the half line again exists globally and tends to the unique steady state. It is helpful in getting our results that the problem turns out to be self-adjoint so that we are able to employ an energy. Such a tool is not available for the power case which suffers additionally from the nonlinearity blowing up at a finite value of the dependent variable.

In Sec. 4, some numerical solutions for both the exponential and power-law cases are carried out. The simulations for the exponential are consistent with the results of Sec. 3. Those for the power-law case also indicate global existence, strongly suggesting that similar results to those for the exponential should apply.

2. Background and derivation of the problem

Problems of thermo-viscous flow and of thermo-viscoelastic flow in channels have previously been seen to lead to non-local parabolic equations [1]. These models have held in bounded domains, corresponding to cross-sections of the channels. Conditions under which such problems can exhibit some form of blow-up, possibly corresponding to shear-band formation, have been investigated in a number of papers [1], [2], [3], [4], [5], [15], [16]. A related problem of thermo-plastic flow arises from the consideration of *linear friction welding.* In this process two metal workpieces are forced to slide against each other, in a rubbing motion, while also being pressed together. The rubbing leads to heat generation, consequent softening near the workpieces' adjoining surfaces, and plastic flow. Viscous dissipation continues heat generation within a thin softened layer where the two workpieces merge. The forcing together of the workpieces results in an additional, but rather smaller, squeezing motion, with the workpieces moving slowly towards each other, and some material – including impurities on the original surfaces – being expelled from the sides as "flash". Fig. 1 gives a sketch of the procedure. After a time the process is halted and cooling of previously warmer regions results in a weld between the workpieces.

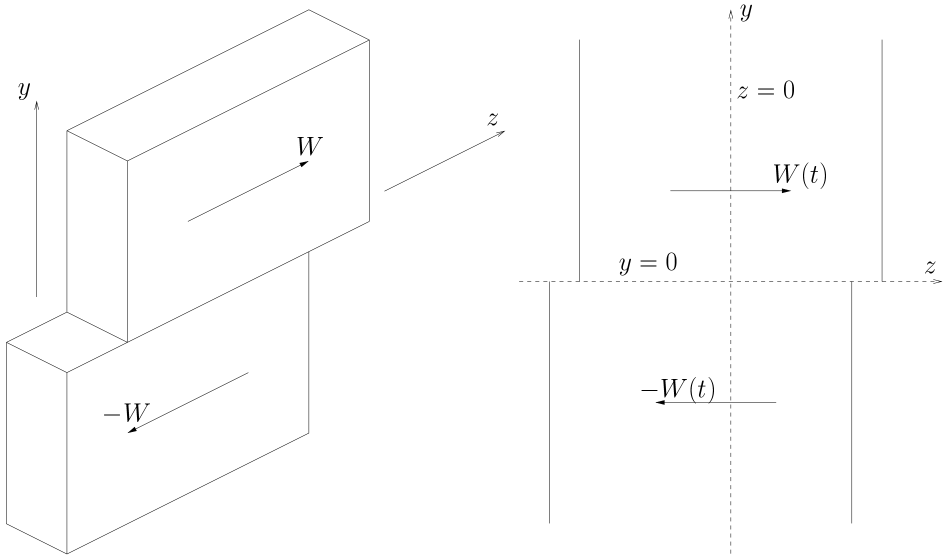

FIGURE 1. Linear friction welding.

In [6] and [13], the process has been modelled by assumimg that the forced sliding velocities of the workpieces, $\pm W(t)$, are square waves with $W(t) = \pm W_e$. This means that the sliding speed is always W_e. This, combined with an assumption that in the soft layer the corresponding and oscillating sliding velocity dominates the non-oscillatory squeezing velocity, means that the rate of heat generation within the layer is, to leading order, independent of time. The latter assumption will hold provided, roughly speaking, that the sliding speed W_e is large enough in comparison with the squeezing force. Of course other material constants and operating parameters will be involved.

Another key part of the model in [6], [13], see also [9], is the thinness of the soft layer, which can then be looked at using lubrication theory, subject to matching conditions with outer problems applying to the cooler parts of the workpieces which move as simple rigid bodies. Fig. 2 indicates some of the key features of the model. If the sides of the workpieces have low Biot number, that is, the heat transfer between them and their surroundings is poor, and the oscillations have small amplitude, temperature will be, to leading order, independent of position, z, along the weld.

All this allows the process to be represented by an essentially one-dimensional mathematical model. The squeezing velocity along the weld, the z direction, $\bar{w}$, is proportional to z, measured from the centre of the weld, but on writing $\bar{w} = zw^*$, the new variable w^* is independent of z. The flow in the soft layer is dominated by one-dimensional, plastic, sliding flow, and is modelled in the same way as the shear-band models of [1], [2]. The energy equation for temperature, however, requires a matching condition, with y, the distance from the plane of symmetry along the weld, large compared with h, the width of the soft

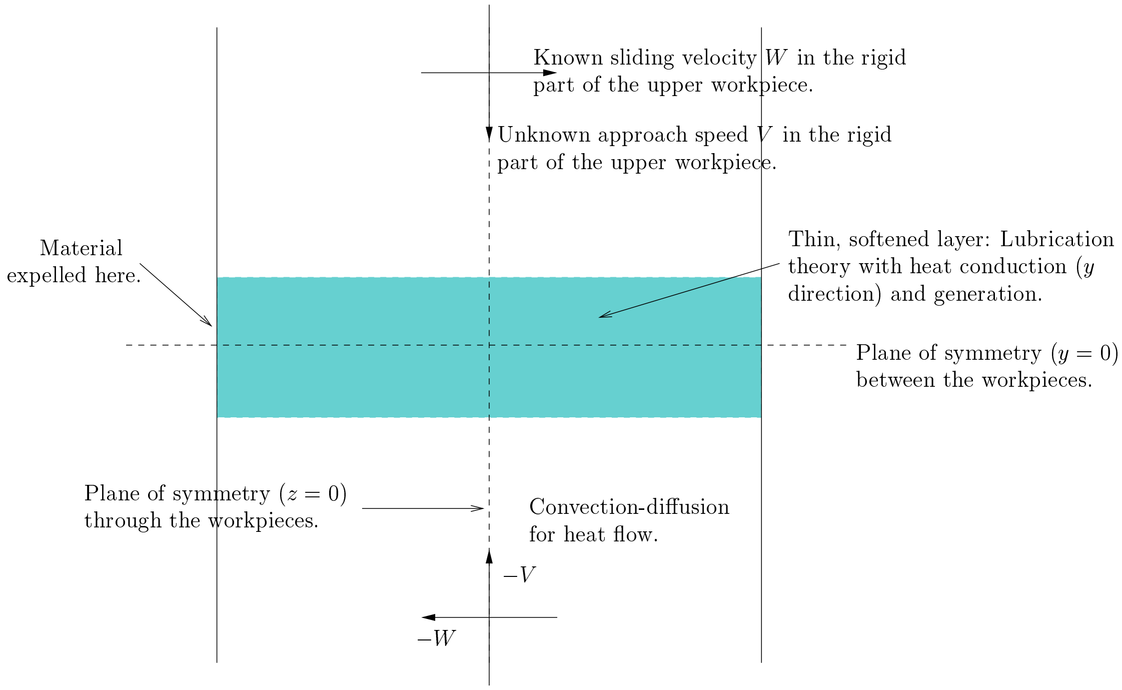

FIGURE 2. Linear friction welding.

layer. (More accurately, h is the length scale across this thin, soft layer.) The matching is with an outer problem for temperature, in the the rigid parts of the workpieces, which gives a value for the temperature gradient in terms of the approach speed V. Thus, looking at the inner problem which applies over a length scale of h in the thin, soft zone, a temperature gradient is applied at infinity – more accurately, for $y/h \gg 1$. This contrasts with the shear-band models, which have boundary conditions imposed for temperature at the surfaces which are a finite distance apart.

A clear problem with the model as outlined above and discussed more fully in [9], [13], is that of course $W(t)$ will not be of constant size; variation something like $W(t) = W_e \cos \Omega t$ is much more reasonable. A preliminary investigation has suggested possible bad behaviour as $W(t)$ passes through 0, when the relative sliding is stopped for an instant, with certain quantities becoming infinite, [13]. The modelling was based upon a quasi-steady theory for temperature in the inner region; this approximation will fail for times sufficiently close to the zeros of $W(t)$. In particular, where time t satisfies $|t-(n+\frac{1}{2})\pi/\Omega| = \mathrm{O}(h^2/D)$, where D is the thermal diffusivity, for any integer n, the time derivative of temperature, T_t, is clearly important and should not be neglected.

In this paper we discuss a model problem which might serve to give some guidance as to the behaviour of the solution to the inner problem in such time regimes. A key sub-problem for present purposes is then the energy equation for temperature T:

$$\rho c T_t = k T_{yy} + \tau w_y \, , \tag{2.1}$$

where ρ is density, c is specific heat, k is thermal conductivity (so $D = k/\rho c$), τ is shear stress which is *a-priori* unknown and w is the velocity in the z direction (along the weld, *i.e.* along the soft layer). The density, specific heat and thermal conductivity are all regarded as being constant. It should be noted in this equation that even if there had been significant temperature variation with z, the assumption of thinness of the layer, *i.e.* that $h \ll L$ if L is the width of a workpiece, would ensure that the T_{zz} term would be negligible compared with the T_{yy} one. Also because of the thinness of the layer, the convective term, involving vT_y if v is the velocity in the y direction, can be neglected from (2.1). The last term in (2.1), τw_y, represents dissipation and acts as a body heat source. Equation (2.1) is taken to hold for $0 < y < \infty$, with an assumed symmetry condition holding along the central plane,

$$T_y = 0 \quad \text{on} \quad y = 0\,,$$

and a matching condition at infinity,

$$T_y \to -A \quad \text{as} \quad y \to \infty\,.$$

Matching with the outer region should be expected to provide a rather stronger condition, $T \sim -Ay + T^*$ for $y \to \infty$, and some results appearing in Sec. 3 will require such a hypothesis.

The flow turns out to be 'slow', in the sense that inertial terms can be neglected from momentum equations, as is usual in lubrication theory. One of these equations then becomes simply

$$\tau_y = 0$$

$$\text{so} \quad \tau = \tau(t)\,.$$

The material is, in the lubricating layer, non-Newtonian and undergoes both shear thinning and thermal softening so that a relevant constitutive equation is of the form

$$\tau = F(T)|w_y|^{a-1}w_y = F(T)|w_y|^a \text{sign}\,(w_y)\,, \tag{2.2}$$

with $F(T)$ a decreasing function of temperature and a is typically 1/4. (Many metals have exponent a close to this value, although other types of material have different power laws [6], [10], [12].) Symmetry is again applied so that

$$w = 0 \quad \text{on} \quad y = 0\,,$$

while the imposed sliding motion gives

$$w \to W(t) \quad \text{as} \quad y \to \infty\,.$$

Equation (2.2) can be rewritten as $w_y = F(T)^{-1/a}|\tau|^{1/a}\text{sign}\,\tau$ which can be integrated to give

$$W = \int_0^\infty w_y\, dy = (\text{sign}\,(\tau))|\tau|^{1/a} \int_0^\infty F(T)^{-1/a} dy$$

$$\text{so} \quad \tau = W(t)^a \text{sign}\,(W(t)) \Big/ \left(\int_0^\infty F(T)^{-1/a} dy\right)^a\,.$$

Substitution of this into (2.1) leads to

$$\rho c T_t = kT_{yy} + W(t)^{a+1}F(T)^{-1/a} \Big/ \left(\int_0^\infty F(T)^{-1/a} dy\right)^{(1+a)}\,,$$

or

$$T_t = DT_{yy} + g(t)f(T) \Big/ \left(\int_0^\infty f(T) dy\right)^{(1+a)}\,, \tag{2.3}$$

where $g(t)$ is regarded as specified and $f(T) = F(T)^{-1/a}$ is an increasing function of temperature.

One further aspect of the problem must be mentioned. Because we are looking at what is really a thin zone, temperature T varies little in comparison with what it does in the outer region. Under various circumstances it is then possible to approximate $F(T)$ by a simple function. In the literature, [10], the temperature-dependent factor is often taken be of Arrhenius type, const.$\times \exp(T_a/T)$ for some activation temperature T_a, but in [6], because temperature can approach that of melting, T_m, a product of an Arrhenius term and a linear term vanishing at T_m was taken to be more appropriate so $F(T) =$ const.$(T - T_m)\exp(T_a/T)$.

The report [6] concentrated on a typical case of a 'hard' material, meaning that the operating conditions and material properties are such that T lies extremely close to T_m and $F(T)$ can then be approximated by a linear function. A partially scaled version of (2.3) is then

$$u_t = u_{xx} - G(t)u^{-p} \bigg/ \left(\int_0^\infty u^{-p} dy\right)^{1+1/p} \qquad \text{for } 0 < x < \infty\,,$$

$$\text{with} \qquad u_x = 0 \text{ on } x = 0 \quad \text{ and } \quad u_x \to 1 \text{ as } x \to \infty\,;$$

here $p = 1/a$ (≈ 4) and we have written $T = T_m - \text{const.} \times u$.

As a first step in trying to see if this is a reasonable model, the corresponding autonomous problem, with G replaced by a constant should be investigated. This is done numerically in Sec. 4.

In [9], [13], both 'hard' and 'soft' materials were considered. In the latter case, which is again subject to having suitable material constants and operating conditions, T does not get close to T_m, in contrast to the former one. Now, for T_a sufficiently large, it is the Arrhenius term which varies significantly in the soft layer and F can be approximated by a simple exponential, as is done in high-activation-energy asymptotics, for instance in thin-flame theory. Now the simplified problem is, with suitable scaling,

$$u_t = u_{xx} + G(t)e^{u} \bigg/ \left(\int_0^\infty e^{u} dy\right)^{1+a} \qquad \text{for } 0 < x < \infty\,,$$

$$\text{with} \qquad u_x = 0 \text{ on } x = 0 \quad \text{ and } \quad u_x \to -1 \text{ as } x \to \infty\,.$$

Again, as a first step, we wish to investigate the autonomous problem. In this particular equation we can, by using suitable rescaling, replace G by 1. Numerical experiments are again carried out in Sec. 4. Because of the structure of this problem, for example we see, Sec. 3, that it admits a suitable energy, we are able to prove that the solution exists for all time and tends, uniformly on compact intervals, to the unique steady state. This is strongly indicative of the non-autonomous model being a well-posed and sensible mathematical problem and of being a good model for the local behaviour of temperature and flow in these thin, short-time regimes occurring in linear friction welding. A fuller model, linking the other time regimes and the outer problem to this short-time, local-in-space problem, is to be the subject of a future paper.

3. The exponential case

In this case the time-dependent problem is written in the form

$$u_t = u_{xx} + \frac{e^u}{\left(\int_0^\infty e^u dx\right)^{1+a}},\ 0 < x < \infty,\ t > 0,\ a > 0, \tag{3.1}$$

$$u_x(0,t) = 0, \quad u_x(x,t) \to -1 \text{ as } x \to \infty,\ t > 0, \tag{3.2}$$

$$u(x,0) = u_0(x),\ 0 < x < \infty. \tag{3.3}$$

We consider $u_0(x)$ which satisfies the compatible conditions

$$u_0'(0) = 0, \quad \text{and} \quad u_0'(x) \to -1 \quad \text{as} \quad x \to \infty, \tag{3.4}$$

and the monotonicity condition

$$u_0'(x) < 0,\ 0 < x < \infty.$$

This condition implies, via the maximum principle, that $u_x(x,t) < 0$ for $0 < x < \infty$ and $t > 0$. For Lemma 3.3, at least, we need the stonger condition that $u_0(x) + x$ is bounded.

Theorem 3.1. *Problem (3.1)-(3.3) has a global-in-time solution.*

Proof. We consider a self-similar solution of the problem

$$v_t = v_{xx}\,,\ 0 < x < \infty,\ t > 0,$$

$$v_x(0,t) = 0\,, \quad v_x(x,t) \to -1 \text{ as } x \to \infty,\ t > 0,$$

of the form $v(x,t) = -t^{1/2}V(\eta)$ for $\eta = xt^{-1/2}$ where $V(\eta)$ satisfies the problem

$$V'' + \frac{\eta}{2}V' - \frac{1}{2}V = 0,\ \ V'(0) = 0,\ \ V(\eta) \sim \eta,\ \eta \to \infty.$$

Then for c sufficiently large negative, $c + v$ is lower solution of problem (3.1)-(3.3) and so

$$u \geq c - t^{1/2}V(Rt^{-1/2}) \quad \text{for every} \quad 0 \leq x \leq R. \tag{3.5}$$

Moreover the function $\theta(x,t)$ satisfying the problem

$$\begin{aligned}
\theta_t &= \theta_{xx} + \frac{e^u}{\left(\int_0^R e^u dx\right)^{1+a}},\ 0 < x < R,\ t > 0,\ a > 0,\\
\theta_x(0,t) &= 0, \quad \theta_x(R,t) = 0,\ t > 0,\\
\theta(x,0) &= u_0(x),\ 0 < x < R,
\end{aligned}$$

is an upper solution of problem (3.1)-(3.3) restricted in the interval $[0,R]$ for every $R > 0$, hence $u(x,t) \leq \theta(x,t)$ for $0 \leq x \leq R$.

Setting $h(x,t) = \frac{e^u}{\left(\int_0^R e^u dx\right)^{1+a}}$ we have

$$\int_0^R h(x,t)\,dx = \frac{1}{\left(\int_0^R e^u\,dx\right)^a} \leq \frac{1}{\left(e^c \int_0^R e^{-t^{1/2}V(R\,t^{-1/2})}\,dx\right)^a} \leq R^{-a}\,e^{-a\,c}\,e^{a\,t^{1/2}\,V(R\,t^{-1/2})}$$

and so

$$u(x,t) \leq \theta(x,t) \leq B_1(R) + B_2(R)\,t^{1/2}\,e^{a\,V(R\,t^{-1/2})\,t^{1/2}},\quad 0 \leq x \leq R,$$

for every $R > 0$. The latter yields that a finite-time blow-up does not occur taking also into account that $u(x,t)$ is decreasing in x. □

The associated steady-state problem of (3.1)-(3.3) is

$$w'' + \frac{e^w}{\left(\int_0^\infty e^w dx\right)^{1+a}} = 0,\ 0 < x < \infty,\ w'(0) = 0,\ w'(x) \to -1 \text{ as } x \to \infty, \tag{3.6}$$

and can be solved exactly to find $w(x) = \ln\left(\frac{1}{2}\,\text{sech}^2\frac{x}{2}\right)$. Note also that $\int_0^\infty e^w dx = 1$.

Setting $z(x,t) = u(x,t) - w(x)$ we arrive at the problem

$$z_t = z_{xx} + \frac{e^u}{\left(\int_0^\infty e^u dx\right)^{1+a}} - \frac{e^w}{\left(\int_0^\infty e^w dx\right)^{1+a}},\ 0 < x < \infty,\ t > 0,\ a > 0, \tag{3.7}$$

$$z_x(0,t) = 0, \quad z_x(x,t) \to 0 \text{ as } x \to \infty,\ t > 0, \tag{3.8}$$

$$z(x,0) = z_0(x) = u_0(x) - w(x). \tag{3.9}$$

To obtain some a priori estimates in order to study the long-time behaviour of this problem we consider the functional

$$J(t) = \frac{1}{2}\int_0^\infty z_x^2 dx + \frac{1}{a}\left(\left(\int_0^\infty e^w e^z dx\right)^{-a} + a\int_0^\infty e^w z\,dx - 1\right).$$

The functional $J(t)$ defines a local semiflow in the sense that $\frac{dJ(t)}{dt} \leq 0$. Indeed,

$$\begin{aligned}
\frac{dJ(t)}{dt} &= \int_0^\infty z_x\,z_{xt}\,dx - \frac{\int_0^\infty e^u\,u_t\,dx}{\left(\int_0^\infty e^u dx\right)^{1+a}} + \int_0^\infty e^w\,z_t\,dx\\
&= -\int_0^\infty (u_{xx} - w'')\,u_t\,dx - \frac{\int_0^\infty e^u\,u_t\,dx}{\left(\int_0^\infty e^u\,dx\right)^{1+a}} + \int_0^\infty e^w\,u_t\,dx \quad \text{(due to the boundary conditions)}\\
&= -\int_0^\infty u_t^2\,dx \leq 0.
\end{aligned}$$

Thus

$$J(t) \le J(0) \equiv J_0 = \frac{1}{2}\int_0^\infty (z_0')^2\,dx + \frac{1}{a}\left(\left(\int_0^\infty e^{u_0}\,dx\right)^{-a} + a\int_0^\infty e^w z_0\,dx - 1\right),$$

or equivalently

$$\frac{1}{2}\int_0^\infty z_x^2 dx + \int_0^\infty e^w z\,dx \le J_0 + \frac{1}{a} - \frac{1}{a}\left(\int_0^\infty e^u dx\right)^{-a} \le J_0 + \frac{1}{a} = K.$$

The latter implies that

$$0 \le \frac{1}{2}\int_0^\infty z_x^2\,dx = K_1 \le K - \int_0^\infty e^w\,z\,dx\,. \tag{3.10}$$

Using Cauchy-Shwartz's inequality and relation (3.10) we derive

$$\left(\int_0^r |z_x| dx\right)^2 \le r\int_0^r z_x^2\,dx \le 2\,K_1\,r$$

which yields that

$$|z(r,t) - z(0,t)| \le \int_0^r |z_x|\,dx \le \sqrt{2\,K_1\,r}.$$

Setting $m(t) = z(0,t)$ we obtain

$$m - \sqrt{2\,K_1\,r} \le z(r,t) \le m + \sqrt{2\,K_1\,r} \quad \text{for every} \quad r > 0 \tag{3.11}$$

and so

$$m - \sqrt{2\,K_1 C_0^2} \le \int_0^\infty e^w\,z\,dx \le m + \sqrt{2\,K_1 C_0^2}, \tag{3.12}$$

writing $\int_0^\infty \sqrt{x}\,e^w\,dx = C_0 \approx 1.072$ and recalling that $\int_0^\infty e^w\,dx = 1$.

Hence, from (3.10), we get

$$0 \le K_1 = \frac{1}{2}\int_0^\infty z_x^2\,dx \le K - m + \sqrt{2\,K_1 C_0^2}$$

so

$$(\sqrt{K_1} - \sqrt{2}\,C_0)\sqrt{K_1} \le K - m,$$

or

$$\left(\sqrt{K_1} - \frac{C_0}{\sqrt{2}}\right)^2 \le K - m + \frac{C_0^2}{2}. \tag{3.13}$$

As well as providing an estimate for K_1, we see from this that m is bounded above.

Lemma 3.2. *As $t \to \infty$, $m(t) \to 0$.*

Proof. Because m is bounded above, there are then just three possible types of long-time behaviour of $m(t)$: (i) $m(t) \to -\infty$ as $t \to \infty$, (ii) $m(t)$ oscillates as $t \to \infty$ or (iii) $m(t) \to m^* > -\infty$

In the former case using that $\frac{\partial z}{\partial x} \le 1$ we obtain $z(x,t) - m(t) \le x$ and via estimate (3.13) we finally derive

$$z(x,t) - m(t) \le \min\{x, \sqrt{2\,K_1\,x}\} = \begin{cases} x, & 0 \le x \le 2\,K_1 \\ \sqrt{2\,K_1\,x}, & x \ge 2\,K_1 \end{cases},$$

hence

$$\int_0^\infty e^u\,dx = \int_0^\infty e^{w+z}\,dx \le e^m\left(\int_0^{2K_1} e^{w+x}\,dx + \int_{2K_1}^\infty e^{w+\sqrt{2\,K_1\,x}}dx\right).$$

Writing $M(t) = -u(0,t) = \ln 2 - m(t)$ then relation (3.13) gives

$$K_1 \le M + C\sqrt{M} \quad \text{for } M \text{ sufficiently large,}$$

where C is a positive constant. Then

$$\begin{aligned}\int_0^{2K_1} e^{w+x}\,dx &\le \int_0^{2K_1} e^{C_1}\,dx \le C_2(M+C\sqrt{M}),\\ \int_{2K_1}^{\infty} e^{w+\sqrt{2K_1 x}}dx &\le \int_{2K_1}^{\infty} e^{C_1-x+\sqrt{2K_1 x}}dx = C_3\int_0^{\infty} e^{2K_1\sqrt{1+x_1/2K_1}-2K_1-x_1}dx_1 < C_4,\end{aligned}$$

for C_1 being a constant such that $w(x)-x\le C_1$, and C_2, C_3, C_3 some positive constants, so finally we get that

$$\int_0^{\infty} e^u\,dx < 2e^{-M}\Big(C_2(M+C\sqrt{M})+C_4\Big) < C_5 M e^{-M} \quad \text{for } M \text{ sufficiently large.} \tag{3.14}$$

Setting $g(t)=\big(\int_0^\infty e^u dx\big)^{-(1+a)}$ in view of (3.14) we obtain

$$g(t)\ge C_6 M^{-(1+a)}e^{M(1+a)} \quad \text{for } M \text{ sufficiently large,}$$

for C_6 being some positive constant.

Now take some $M_2 > M_1 > 0$, to be chosen suitably later, so that for some time t_1,

$$u(0,t_1)=-M_1,\ u(0,t)<-M_1 \text{ for } t_1<t<t_1+\delta \text{ and } u(0,t)>-M_2 \text{ for } 0\le t\le t_1+\delta, \tag{3.15}$$

for some $\delta>0$. If (3.15) does not hold then $u(0,t)$ is certainly bounded below and so $m(t)$ is, leading to contradiction.

While $M(t)=u(0,t)\ge -M_2$, $-M_2-x$ is a simple lower solution for u so $u(x,t)>-M_2-x$ for $0\le t\le t_1+\delta$ and $x>0$. On the other hand, relation (3.11) implies that $u(x,t)\ge -M(t)+\ln 2+w(x)-\sqrt{2K_1(M(t))x}$ for $x>0$, $t>0$, with $K_1(M)$ defined by the largest value of K_1 such that (3.13) holds. In particular at $t=t_1$ we have

$$u(x,t)\ge \max\big\{-M_2-x,\ -M_1+\ln 2+w(x)-\sqrt{2K_1(M_1)x}\,\big\}.$$

Also, as long as $M(t)\ge M_1$ (i.e. $u(0,t)\le -M_1$), which is certainly true for $t_1\le t\le t_1+\delta$, we get $g(t)\ge C_6 e^{(1+a)M}M^{-(1+a)}\ge C_6 e^{(1+a)M_1}M_1^{-(1+a)}$ for M_1 large enough. Writing $u=U-M_2$ we have

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}+g(t)e^{-M_2}e^U,\ \ 0<x<\infty,\ \ t>t_1, \tag{3.16}$$

$$\frac{\partial U}{\partial x} = 0 \text{ at } x=0,\ \ U_x\to -1 \text{ as } x\to\infty,\ \ t>t_1, \tag{3.17}$$

with

$$\begin{aligned}U(x,t_1) &\ge \max\big\{-x,\ M_2-M_1+\ln 2+w(x)-\sqrt{2K_1(M_1)x}\,\big\}\\ &\ge \max\big\{-x,\ C_7\sqrt{M_1}(1-\sqrt{x})+w(x)\big\}\\ &\ge \begin{cases} C_7\sqrt{M_1}(1-\sqrt{x}), & 0\le x<1\\ -x, & x\ge 1\end{cases},\end{aligned} \tag{3.18}$$

for C_7 being a positive constant and choosing $M_2=M_1+C_7\sqrt{M_1}-w(1)$ for M_1 large enough. Also, $g(t)e^{-M_2}\ge C_6 e^{(aM_1-C_7\sqrt{M_1})}M_1^{-(1+a)}\ge C_8 e^{C_9 M_1}$ for M_1 sufficiently large.

It is easily seen that

$$W(x)\equiv \max\{-x,\ k(1-x^2)-1\} = \begin{cases} k(1-x^2)-1, & 0\le x<1\\ -x, & x\ge 1\end{cases},$$

is a lower solution of problem (3.16)-(3.18) for $t\ge t_1$, while $U(0,t)\le C_7\sqrt{M_1}$, provided that

- *(i)* $k>1$ (to ensure that $U(0,t)\ge W(0)>0$, i.e. $u(0,t)>-M_2$, and that $W(x)$ is well-defined, i.e. $k(1-x^2)-1+x\ge 0$ for $0\le x<1$),
- *(ii)* $k\le \frac{C_7\sqrt{M_1}}{4}$ (to ensure that $W(x)\le U(x,t)$ at $t=t_1$, i.e. $k(1-x^2)-1\le C_7\sqrt{M_1}(1-\sqrt{x})$ for $0\le x<1$),

- (iii) $k \le \frac{C_8}{2} e^{C_9 M_1 - 1}$(to ensure that $W'' + g e^{-M_2} e^W \ge 0$).

Taking $k = \frac{C_7}{4}\sqrt{M_1}$, and M_1 large enough, then $(i) - (iii)$ all hold, and W is a lower solution to (3.16)-(3.18) for $t \ge t_1$. Thus $U(0,t) > W(0,t) > 0$, i.e. $u(0,t) > -M_2$ as long as $u(0,t) \le -M_1$. Consequently, we have $u(0,t) > -M_2$ for every $t > 0$ which contradicts the hypothesis (i), that $m(t) \to -\infty$ as $t \to \infty$.

In the second case, when $m(t)$ oscillates, there exist $m_a < m_b$ and sequences of time $t_1 \le \widehat{t_1} \le t_2 \le \widehat{t_2} \le ...$ such that $m(t_n) = m_a,\ m(\widehat{t}_n) = m_b$ and $m_a < m(t) < m_b$ for $t_n < t < \widehat{t}_n$. Then for $t_n < t < \widehat{t}_n$ we obtain

$$\begin{aligned}\int_0^\infty e^u\,dx &\ge \frac{1}{2}e^m \int_0^\infty \operatorname{sech}^2\frac{x}{2}\, e^{-\sqrt{2K_1 x}}\,dx \ge \frac{1}{2}e^m \int_0^\infty e^{-x-\sqrt{2K_1 x}}\,dx\\ &\ge \frac{1}{2}e^{m_a}\int_0^\infty e^{-x-\sqrt{2\widehat{C}_2 x}}\,dx = A > 0,\end{aligned}$$

where $\widehat{C_1} = \left(\sqrt{K - m_a + \frac{C_0^2}{2}} + \frac{C_0}{\sqrt{2}}\right)^2$ and A is a constant.

Also, in such time intervals, $m(t)$ satisfies

$$\frac{dm}{dt} = \frac{d}{dt}u(0,t) \le \frac{e^{u(0,t)}}{\left(\int_0^\infty e^u\,dx\right)^{1+a}} = \frac{e^m}{2\left(\int_0^\infty e^u\,dx\right)^{1+a}} \le e^{m_b} A^{-(1+a)} = \widehat{C}_2,$$

which after integration implies $\widehat{t}_n - t_n \ge \delta = \frac{m_b - m_a}{\widehat{C}_2} > 0$.

Using the estimate (3.11) for $z(x,t)$ we can obtain a lower bound for the functional $J(t)$. Indeed, taking also into account (3.12) and (3.13) we derive

$$J(t) \ge \int_0^\infty e^w\, z\,dx - \frac{1}{a} \ge m - \sqrt{\frac{K_1 C_0^2}{2}} - \frac{1}{a} \ge m_a - \frac{\sqrt{C_1 C_0^2}}{\sqrt{2}} - \frac{1}{a}.$$

Thus, taking a subsequence if necessary, there exists a sequence $\tilde{t}_n \in (t_n, \widehat{t}_n)$ s.t. $\dot{J}(\tilde{t}_n) = \max_{[t_n,\widehat{t}_n]} \dot{J}(t)$ with $\dot{J}(\tilde{t}_n) \to 0$ as $n \to \infty$, where $\dot{} = \frac{d}{dt}$, and $m(\tilde{t}_n) \to m^*$ as $n \to \infty$. This implies that the sequence $u(\cdot,\tilde{t}_n)$ is uniformly bounded in $C([0,R])$ for every $R > 0$. Using Schauder-type estimates we obtain that $u(\cdot,\tilde{t}_n)$ is uniformly bounded in $C^1([0,R]) \cap C^2((0,R))$ as well. Therefore, there is a subsequence denoted again by $u(x,\tilde{t}_n)$ and a function $\widehat{w}(x)$ such that

$$u(\cdot,\tilde{t}_n) \to \widehat{w}(\cdot) \quad \text{as} \quad n \to \infty \quad \text{in} \quad C^1([0,R]) \cap C^2((0,R)) \quad \text{for every} \quad R > 0.$$

Taking now the inner product in $L^2([0,R])$ of (3.1) with an arbitrary function $\xi \in H_0^1([0,R])$ for $t = \tilde{t}_n$ we derive

$$\int_0^R u_t(x,\tilde{t}_n)\,\xi(x)\,dx = -\int_0^R u_x(x,\tilde{t}_n)\,\xi'(x)\,dx + \int_0^R \frac{e^{u(x,\tilde{t}_n)}}{\left(\int_0^\infty e^{u(y,\tilde{t}_n)}\,dy\right)^{1+a}}\,\xi\,dx, \tag{3.19}$$

where ξ' is the distributional derivative of ξ.

Passing to the limit as $\tilde{t}_n \to \infty$ of (3.19), taking also into account,

$$\int_0^R u_t^2(x,\tilde{t}_n)\,dx \to 0 \quad \text{as} \quad n \to \infty \quad \text{for every} \quad R > 0,$$

and Lebesgue's dominated convergence theorem we deduce that

$$-\int_0^R \widehat{w}'(x)\,\xi'(x)\,dx + \int_0^R \frac{e^{\widehat{w}(x)}}{\left(\int_0^\infty e^{\widehat{w}(y)}\,dy\right)^{1+a}}\,\xi\,dx = 0.$$

Since $\widehat{w}(x) \in C^1([0,R]) \cap C^2((0,R))$ for every $R > 0$, we obtain that $\widehat{w}(x)$ concides with the unique classical steady-state solution, hence $m^* = 0$. This leads to a contradiction since under the hypothesis that $m(t)$ oscillates we can always find $m_a < m_b$ with $0 \notin (m_a, m_b)$. Using the same arguments we can also rule out case (iii), that $m(t) \to m^* \ne 0$. Thus we finally deduce that $m(t) \to 0$ as $t \to \infty$. □

We claim that $u(x,t) \to w(x)$ as $t \to \infty$ for every $0 < x < \infty$. To prove this we need the following

Lemma 3.3. *The difference between the solution u and equilibrium w, $z(x,t) = u(x,t) - w(x)$, is uniformly bounded.*

Proof. Using the fact that $u(0,t) \to w(0) = -\ln 2$ as $t \to \infty$ we easily get that $z(x,t)$ is bounded from below, i.e. $u(x,t) \geq M_l - x$ for $M_l \leq \min\{\inf_{t>0} u(0,t), \inf_{x>0}(u_0(x)+x)\} < 0$.

To obtain an upper bound for $u(x,t)$ we appeal to intersection comparison arguments. Consider a related steady-state solution $w(x;\lambda,R)$ satisfying the equation $w'' + \lambda e^w = 0$, centred on $R \geq 0$, i.e. $w(x;\lambda,R) = \ln\left(\frac{1}{2\lambda}\text{sech}^2\left(\frac{x-R}{2}\right)\right)$, for some $\lambda > 0$. For any λ we can take R large enough so that $u_0(x)$ crosses $w(x;\lambda,R)$ exactly once, using the fact that $u'_0 < 0$. With such an R, $w(.,\lambda,R) > u$ at infinity and, because $u_x < 0$, if a second crossing of occurs at some time it must appear at some time t_a with $u(x,t_a)$ tangent to $w(x,\lambda,R)$ at some point $x_a > R$.

There are then two possibilities:
(a) $u(x,t_a)$ crosses $w(x,\lambda,R)$ at some $x < R$ and then $u(x,t_a)$ touches $w(x,\lambda,R)$ from below at $x_a > R$;
(b) at some time, u crosses $w(.,\lambda,R)$ at $x = R$.

In the former case we should have $\lambda \leq (\int_0^\infty e^u\,dx)^{-(1+a)}$ at $t = t_a$. On the other hand with $u(x,t)$ bounded below, e.g. $u(x,t) \geq M_l - x$, a choice of

$$\lambda \geq \lambda_1 = \left(\int_0^\infty e^{M_l - x}\,dx\right)^{-(1+a)} \geq \left(\int_0^\infty e^u\,dx\right)^{-(1+a)},$$

ensures this cannot happen. Notice that λ_1 is independent of R.

In case (b) we proceed as following. From the definition of $J(t)$ we deduce that

$$J(t) > \frac{1}{2}\int_0^R z_x^2\,dx + \int_0^\infty e^w\,z\,dx - \frac{1}{a} \quad \text{for every} \quad R > 0. \tag{3.20}$$

¿From the lower bound on $u(x,t)$, we derive

$$\int_0^\infty e^{w(x)}\,z(x)\,dx \geq \int_0^\infty e^{w(x)}\,(M_l - x - w(x))\,dx = K_2.$$

Furthermore at $t = t_b$ we have

$$\begin{aligned}
\int_0^R z_x^2\,dx &\geq \frac{[z(R,t_b) - z(0,t_b)]^2}{R} \\
&= \frac{[u(R,t_b) - w(R) - m(t_b))]^2}{R} \\
&= \frac{\left[-\ln 2\lambda + \ln\left(2\cosh^2\frac{R}{2}\right) - m(t_b)\right]^2}{R} \\
&\geq \left[\ln e^R - \ln 4\lambda - M\right]^2/R \quad \text{for sufficiently large } R, \text{ writing } M = \sup_{t>0} m(t) \\
&\geq R - 2(\ln 4\lambda + M).
\end{aligned}$$

Then using (3.20) we obtain, taking $R \geq 2\left[J(0) - K_2 + \frac{1}{a} + \ln 4\lambda + M\right]$,

$$J(t_b) > \frac{R}{2} - (\ln 4\lambda + M) + K_2 - \frac{1}{a} \geq J(0),$$

which cannot happen.

Thus $u(x,t)$ intersects $w(x;\lambda,R)$ at most once for every $t > 0$ and so there should be some constant M_u such that $u(x,t) \leq M_u - x$ for $0 < x < \infty$ and $t > 0$, *i.e.* z is uniformly bounded. □

Lemma 3.4. *There exists a constant $C > 0$ such that $|\ddot{J}(t)| < C$ for every $t > 0$.*

Proof. Indeed we have that

$$|\ddot{J}(t)| = 2\left|\int_0^\infty z_t z_{tt} dx\right| \leq 2\int_0^\infty |z_t||z_{tt}|dx,$$

so it suffices to show that

$$\int_0^\infty |z_t||z_{tt}|dx \le C < \infty.$$

Since $M_l - x \le u(x,t) \le M_u - x$ for $0 < x < \infty$, we have that $H(x,t) = e^u/(\int_0^\infty e^u dx)^{1+a}$ is uniformly integrable in $(0,\infty)$ for every $t > 0$. But

$$u_{xx}(x,t) = \int_0^\infty G_{xx}(x,\zeta,t)\, u_0(\zeta)\, d\zeta + \int_0^t \int_0^\infty G_{xx}(x,\zeta,t-\tau)\, \big(H(\zeta,\tau) - H(x,\tau)\big)\, d\zeta d\tau, \tag{3.21}$$

where $G(x,t)$ is the Green function of the heat equation in $(0,\infty)$ satisfying the boundary conditions $G_x(0,t) = 0$ and $G_x(x,t) \to 0$ as $x \to \infty$. Using the smoothing effect of the Green function and the uniform integrability of $H(x,t)$ in $(0,\infty)$, we obtain from (3.21) that $u_{xx}(x,t)$ is uniformly integrable in the half line for every $t > 0$ and going back to the equation (3.1) we deduce that the same holds for $u_t(x,t)$. Since $z_t = u_t$ we derive that z_t is bounded, for $t \ge \epsilon > 0$, as well as integrable in $(0,\infty)$.

Using now a bootstrap argument we can prove that $\int_0^\infty |H_t(x,t)| < K_4 < \infty$ and via relation

$$u_{txx}(x,t) = \int_0^\infty G_{xx}(x,\zeta,t)\, u_t(\zeta,0)\, d\zeta + \int_0^t \int_0^\infty G_{xx}(x,\zeta,t-\tau)\, (H_t(\zeta,\tau) - H_t(x,\tau))\, d\zeta d\tau,$$

we obtain that $\int_0^\infty |u_{txx}(x,t)|dx < K_5 < \infty$. Therefore from

$$u_{tt}(x,t) = u_{txx}(x,t) + H_t(x,t)$$

we deduce that $u_{tt}(x,t)$ is uniformly integrable in the half line for every $t > 0$ and so is $z_{tt}(x,t)$. Hence we derive

$$|\ddot{J}(t)| \le 2\int_0^\infty |z_t||z_{tt}|dx < C < \infty,\ \ t > 0. \tag{3.22}$$

$\square$

Theorem 3.5. *The solution $u(x,t)$ of problem (3.1)-(3.3) converges as $t \to \infty$ to the unique steady-state solution $w(x)$.*

Proof. First we prove that

$$\lim_{t\to\infty} \dot{J}(t) = 0. \tag{3.23}$$

We assume that there exists a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to \infty$ as $n \to \infty$, such that

$$\lim_{n\to\infty} \dot{J}(t_n) = -\lim_{n\to\infty} \int_0^\infty u_t^2(x,t_n)dx \to -c < 0 \tag{3.24}$$

and we derive a contradiction.

Due to (3.24) there exists N such that $\dot{J}(t_n) < -2c/3$, for $n \ge N$. Using $\ddot{J}(t) < K$, we obtain $\dot{J}(t) < -c/3$ for $t_n \le t \le t_n + c/3K$ and $n \ge N$. The latter yields $J(t_n) \to -\infty$ as $n \to \infty$ leading to a contradiction.

Now since $z(x,t)$ is uniformly bounded in every interval $[0,R]$, $R > 0$, we obtain via a Schauder-type estimate that $z(x,t)$ belongs to $C^{2+q,1+q/2}([0,R]\times(0,\infty))$, for some $q > 0$, and for every $R > 0$.

The latter implies that there exists a sequence $t_n \to \infty$ and a function $\psi(x)$ such that

$$||z(\cdot,t_n) - \psi(\cdot)||_{C^1([0,R])\cap C^2((0,R))} \to 0 \ \text{ as } \ n \to \infty, \ \text{ for every } \ R > 0\,,$$

or equivalently that

$$\begin{aligned}\omega(u_0) &= \{\phi \in L^\infty((0,\infty)) : \text{ there exists } t_n \to \infty : ||u(\cdot,t_n;u_0) - \phi(\cdot)||_{C^1([0,\infty))\cap C^2((0,\infty))} \to 0\} \\ &\neq \emptyset.\end{aligned}$$

Now we claim that $\omega(u_0) \subseteq S$, for S the set of the steady states, and hence $\omega(u_0) = \{w\}$, since $S = \{w\}$. Indeed, considering $\phi \in \omega(u_0)$, there is a sequence $(t_n)_{n\in\mathbb{N}}$ with $t_n \to \infty$ as $n \to \infty$, such that $||u(\cdot,t_n) - \phi(\cdot)||_{C^1([0,R])\cap C^2((0,R))} \to 0$, or equivalently $||z(\cdot,t_n) - \psi(\cdot)||_{C^1([0,R])\cap C^2((0,R))} \to 0$ as $n \to \infty$, for $\psi = \phi - w$. By similar arguments as in Lemma 3.2, taking also into account (3.23), we derive $\psi \equiv 0$, hence $\omega(u_0) = \{w\}$. The latter yields the desired result, otherwise, due to the uniform boundedness of z, there must be a subsequence $t_m \to \infty$ such that $||z(\cdot,t_m)-\psi(\cdot)||_{C^1([0,\infty))\cap C^2((0,\infty))} \to 0$ or, equivalently, $||u(\cdot,t_m)-\phi(\cdot)||_{C^1([0,\infty))\cap C^2((0,\infty))} \to 0$ as $m\to\infty$ with $\phi \neq w$, contradicting the above result. □

4. Numerical Results

Because of the fact that the problem is defined in an infinite domain, $[0,\infty)$, an approximated problem is solved numerically in the domain, $[0,b]$, for b large enough, i.e.

$$u_t = u_{xx} + \frac{e^u}{I^{1+a}}, \quad 0<x<b, \quad t>0, \tag{4.1}$$

$$u_x(0,t)=0, \quad u_x(b,t)=-1, \tag{4.2}$$

$$u(x,t)=u_0(x). \tag{4.3}$$

Regarding the term, I, we have

$$I=\int_0^\infty e^u dx = \int_0^b e^u dx + \int_b^\infty e^u dx \ \sim \int_0^b e^u dx + 2e^{-b}, \text{ taking } b \gg 1,$$

assuming that for x large $u\sim w$.

In order to solve numerically problem (4.1) - (4.3), a three-step Crank-Nicolson scheme is used (which is unconditionally stable). Taking a partition of $M+1$ points in $[0,b]$, $0=x_0, x_0+\delta x = x_1, \cdots, x_M=b$ and using a time step δt for a partition in the time interval $[0,T]$ we have

$$\frac{u_j^{i+1}-u_j^{i-1}}{2\delta t} = \frac{1}{2}\frac{u_{j+1}^{i+1}-2u_j^{i+1}+u_{j-1}^{i+1}}{\delta x^2} + \frac{1}{2}\frac{u_{j+1}^{i-1}-2u_j^{i-1}+u_{j-1}^{i-1}}{\delta x^2} + \frac{e^{u_j^i}}{\left(\int_0^b e^{u^i}dx + e^{-b}\right)^{1+a}}. \tag{4.4}$$

The integral $\int_0^b e^{u^i}dx$, where $u^i=(u_1^i,u_2^i,\cdots,u_M^i)^T$, is evaluated in each time step by Simpson's rule. Taking into account the boundary conditions the numerical scheme takes the form

$$Au^{i+1} = Bu^{i-1} + bu^i \tag{4.5}$$

or

$$u^{i+1} = A^{-1}Bu^{i-1} + A^{-1}bu^i,$$

where A, B are $M\times M$ matrices and b $1\times M$ vector.
Numerical results. The first attempt is to investigate if by changing b we notice any difference in the solution.

In Figure (4), $u(x,t)$ is plotted against $x\in[0,b]$ for various values of b together with the steady state solution w. In any case $u(t,x)$ converge to the steady state. Running the program for larger values of t, starting with initial condition $u_0(x)=0$, $u(x,t)$ coincides with w (thick line). The conclusion is that the numerical solution approximates the solution of (3.1-3.3) if b is large enough, e.g. $b\geq 5$.

Now in the next plot, Figure (2), $u(t,0)$ is plotted against time starting with initial condition $u_0(x)=0$ for various values of a. For $a>0$ the solution converges to the steady state while for $a<0$ the solution blows up.

4.1. **The case of $f(u)=u^{-p}$.** For the case that $f(u)=u^{-p}$ no analytical results are available yet for the problem. Therefore in order to get an indication of the behaviour of the solution for this case we present a numerical approximation of the problem.

The problem in this case takes the form

$$u_t = u_{xx} + \frac{(-u)^{-p}}{\left(\int_0^\infty (-u)^{-p}dx\right)^{1+a}}, \ 0<x<\infty, \ t>0, \ a>0, \tag{4.6}$$

$$u_x(0,t) = 0, \quad u_x(x,t)\to -1 \text{ as } x\to\infty, \ t>0, \tag{4.7}$$

$$u(x,0) = u_0(x), \ 0<x<\infty. \tag{4.8}$$

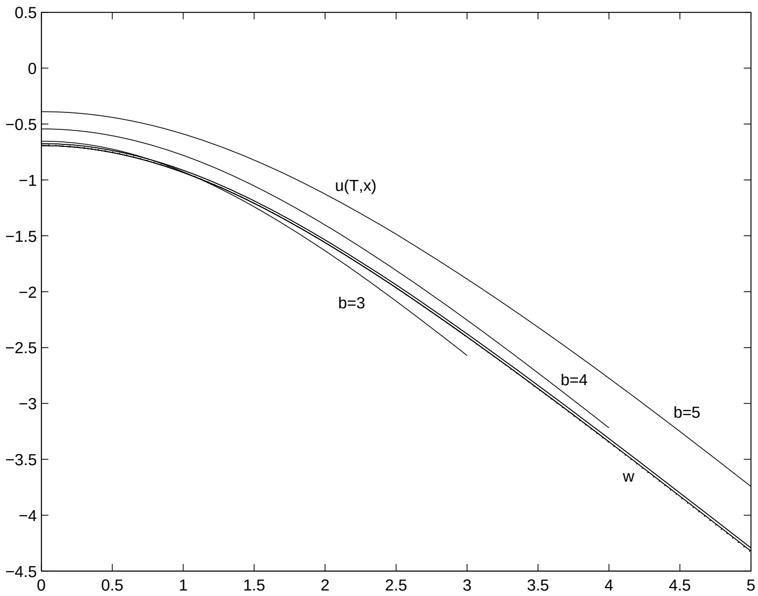

FIGURE 3. *The numerical solution of problem (4.1)- (4.3), for various values of* b *for* $a = 1/2$.

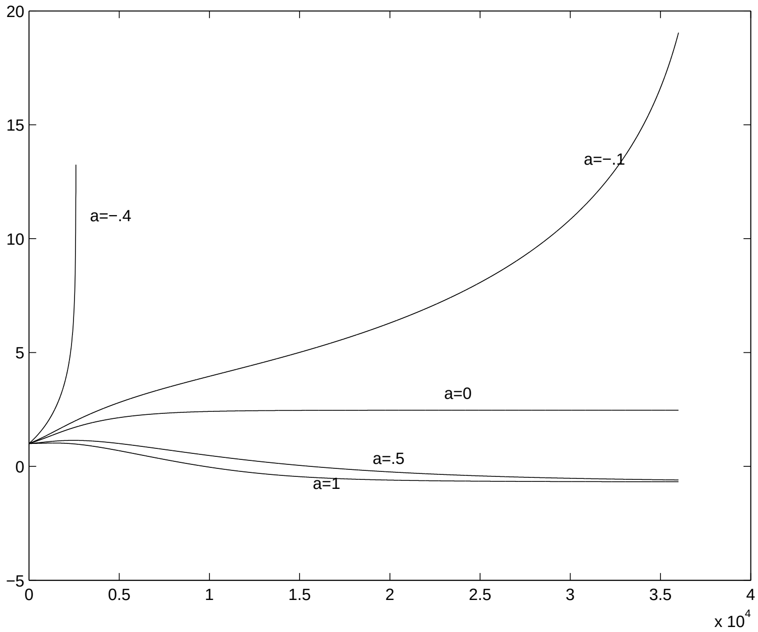

FIGURE 4. *The numerical solution of problem (4.1)- (4.3),* $u(t, 1)$ *is plotted for various values of* $a = -0.4, -0.1, 0, 0.5, 1$ *for* $b = 5$.

An equivalent form of the problem can be given by replacing u by $-u$ and setting $u_x(x,t) \to 1$. Then the problem has the form

$$\begin{aligned} u_t &= u_{xx} - \frac{u^{-p}}{\left(\int_0^\infty u^{-p}\,dx\right)^{1+a}},\ 0 < x < \infty,\ t > 0,\ a > 0, && (4.9)\\ u_x(0,t) &= 0, \quad u_x(x,t) \to 1 \text{ as } x \to \infty,\ t > 0, && (4.10)\\ u(x,0) &= u_0(x),\ 0 < x < \infty. && (4.11) \end{aligned}$$

As for the exponential case we will apply a finite difference scheme to the approximate problem in the domain $0 < x < b$ for b being large enough. The approximate problem is

$$\begin{aligned} u_t &= u_{xx} - \frac{u^{-p}}{\left(\int_0^b u^{-p}\,dx\right)^{1+a}}, \ 0 < x < b, \ t > 0, \ a > 0, & (4.12)\\ u_x(0,t) &= 0, \quad u(b,t) = b \quad t > 0, & (4.13)\\ u(x,0) &= u_0(x), \ 0 < x < \infty. & (4.14)\end{aligned}$$

The finite difference scheme in this case takes the form

$$\frac{u_j^{i+1} - u_j^{i-1}}{2\delta t} = \frac{1}{2}\frac{u_{j+1}^{i+1} - 2u_j^{i+1} + u_{j-1}^{i+1}}{\delta x^2} + \frac{1}{2}\frac{u_{j+1}^{i-1} - 2u_j^{i-1} + u_{j-1}^{i-1}}{\delta x^2} - \frac{(u_j^i)^{-p}}{\left(\int_0^b (u^i)^{-p}dx + \frac{b^{-p+1}}{-p+1}\right)^{1+a}}. \quad (4.15)$$

Again the integral $\int_0^b (u^i)^{-p}dx$, where $u^i = (u_1^i, u_2^i, \cdots, u_M^i)^T$, is evaluated in each time step by Simpson's rule. Taking into account the boundary conditions the numerical scheme takes the form of equation 4.5.

In addition in this case, in order to test the convergence of the scheme (4.15), we have to approximate the steady state solution of the problem,

$$\begin{aligned} w_{xx} &= \frac{w^{-p}}{\left(\int_0^b w^{-p}dx\right)^{1+a}}, \ 0 < x < b, \ a > 0, & (4.16)\\ w_x(0,t) &= 0, \quad w(b,t) = b. & (4.17)\end{aligned}$$

Again using a simple finite difference scheme we have that

$$\frac{w_{j+1} - 2w_j + w_{j-1}}{\delta x^2} = \frac{w^{-p}}{\left(\int_0^b (w)^{-p}dx\right)^{1+a}}. \quad (4.18)$$

This results to a nonlinear algebraic system of the form $Mw = b(w) + c$ which is solved by using a Newton-Raphson iterative scheme.

In Figure (4.1) the numerical solution of the steady-state problem is plotted, for $p = 4$ and for different values of the parameter b. The result indicates that as b increases the numerical solution converges.

In Figure (4.1) the problem (4.12-4.14) is solved numerically in a time interval $[0, T]$ and the solution at time t is plotted against space for different values of the parameter p (solid line). In each case the solution approaches the relevant steady state (dotted line).

Finally in Figure (4.1) the problem (4.12-4.14), with $u(x, 0) = 1$, is solved numerically and the minimum of the solution $u(0, t)$ is plotted against time for different value of the parameter p. In all of the simulations $u(0, t)$ initially decreases but it never reaches zero.

In conclusion these numerical results indicate that the solution u of the problem (4.9-4.11) does not quanch and that converges to the steady state solution.

5. Conclusions

We have seen that for the case of an exponential nonlinearity, the classical results of boundedness, global existence, and convergence to a unique steady state for one-dimensional, non-local equations of the form (1.1) carry over from problems on bounded intervals to the half line. The proofs relied both on the exponential being finite for finite argument and on the existence of an associated energy functional. Both these properties are missing from problems with nonlinearities of the form $|u|^{-p}$. However, numerical results are strongly indicative that solutions, once again, are bounded above, exist globally, and converge to the unique steady state. It is conjectured that equations of the form (1.1), even with an explicit time-dependent factor can therefore be good models for the local behaviour of temperature in problems of linear-friction welding.

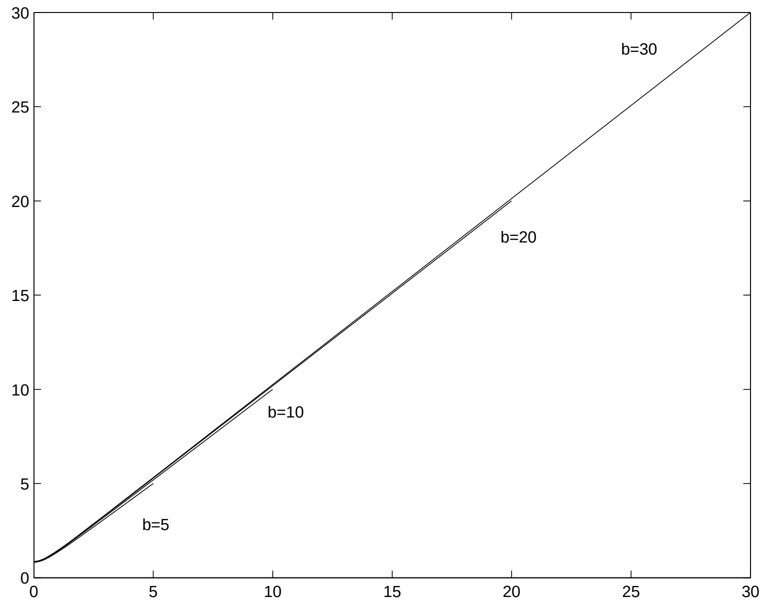

FIGURE 5. *The numerical solution of problem (4.15-4.17), $w(x)$ is plotted for various values of $b = 5, 10, 20, 30$ for $p = 4$.*

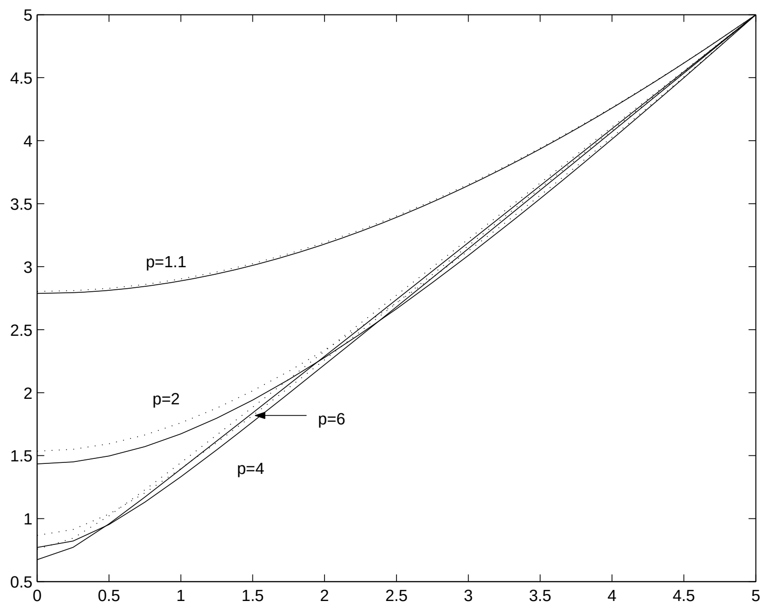

FIGURE 6. *The numerical solution of problem (4.12-4.14), $u(x,t)$ is plotted for various values of $p = 1.1, 2, 4, 6$ for $b = 5$.*

Acknowledgements: The first and the third author have been supported by the grant Pythagoras No. 68/831, Greek Ministry of Education. The project is co-funded by the European Social Fund (75%) and National Resources (25%) - (EPEAEK II) -PYTHAGORAS. The first author was also supported by the Greek State Scholarship Foundation (I.K.Y.) The work of the first author started when he was visiting the Department of Mathematics at Heriot-Watt University. He would like to thank the Department for its hospitality.

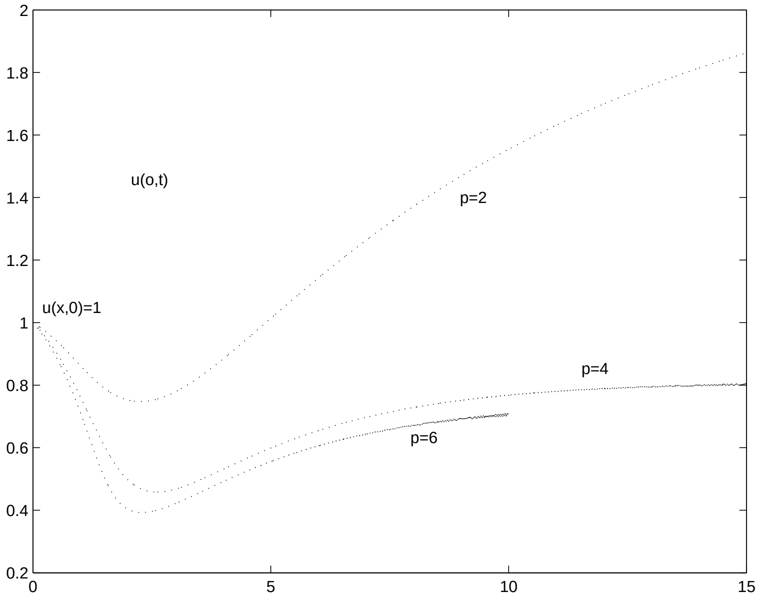

FIGURE 7. *The numerical solution of problem (4.12-4.14), $u(0,t)$ is plotted against time for various values of $p = 2, 4, 6$ for $b = 5$ and $u(x,0) = 1$.*

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