Compact metric spaces and weak forms of the axiom of choice

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Abstract

It is shown that *for compact metric spaces* (X, d) the following statements are pairwise equivalent: *X is Loeb, X is separable, X has a well ordered dense subset, X is second countable* and *X has a dense set $G = \cup\{G_n : n \in \omega\}, |G_n| < \omega$ with $\delta_n = Diameter(G_n) \to 0$*. Further, it is shown that the statement: *Compact metric spaces are weakly Loeb* is not provable in ZF^0, the Zermelo - Fraenkel set theory without the axiom of regularity, and that the *countable axiom of choice for families of finite sets* CAC_{fin} does not imply the statement *Compact metric spaces are separable.*

Keywords: Countable axiom of choice for families of finite sets, Countable axiom of choice, Countable axiom of multiple choice, Compact metric spaces, Loeb metric spaces, Weakly Loeb metric spaces, Separable metric spaces, Second countable metric spaces.

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1 Definitions and introduction

In the Zermelo-Fraenkel set theory with the axiom of choice ZFC a compact metric space is always separable. Without AC this result is not valid, see [4] and [8]. This paper is a continuation of the study of Lindelöf metric spaces having properties like separability, second countability etc and which was initiated in [8]. This time we consider the following notion of a Loeb space introduced by N. Brunner [1]: Let (X, d) be a metric space. Then X is a *Loeb* (*weakly Loeb*) space if there is a choice (multiple choice) function on the family of its non empty closed subsets.

Next we list the statements and choice principles we are going to use in this paper.

1. The *axiom of choice* AC (Form 1 in [5]) : *For every family* $\mathcal{A} = \{A_i : i \in I\}$ *of pairwise disjoint non empty sets there exists a set* $C = \{c_i : i \in I\}$ *such that* $c_i \in A_i$ *for all* $i \in I$.

2. The *countable axiom of choice* CAC (Form 8 in [5]) : AC *restricted to countable families of sets.*

3. CAC_{fin} (Form 10 in [5]) : CAC *restricted to countable families of finite sets.*

4. The *axiom of multiple choice* MC (Form 67 in [5]) : *For every family* $\mathcal{A} = \{A_i : i \in I\}$ *of pairwise disjoint non empty sets there exists a set* $\mathcal{F} = \{F_i : i \in I\}$ *of finite non empty sets such that* $F_i \subseteq A_i$ *for all* $i \in I$.

5. The *countable axiom of multiple choice* CMC (Form 126 in [5]) : MC *restricted to countable families of sets.*

6. Form 131 in [5] : *For every family* $\mathcal{A} = \{A_i : i \in \omega\}$ *of pairwise disjoint non-empty sets, there is a function* f *such that for each* $i \in \omega$, $f(A_i)$ *is a non-empty countable subset of* A_i.

7. Form 9 in [5] : *Every infinite set has a countably infinite subset.*

8. Form 154 in [5] : *The countable Tychonoff product of compact* T_2 *topological spaces is compact.*

9. Form 343 in [5] : *A product of non-empty compact T_2 topological spaces is non-empty.*

10. $(\mathcal{D})$: *Every compact dense-in-itself metric space (X,d) has a dense set $G = \cup\{G_n : n \in \omega\}, |G_n| < \omega$ and $\delta_n = Diameter(G_n) \to 0$.*

11. CMWS : *Every compact metric space (X,d) has a dense subset $\mathcal{D}$ which can be written as a well ordered union of finite sets.*

12. $(\oplus)$: *Every compact metric space (X,d) can be written as a well ordered union of finite sets.*

13. $(\pm)$: *Every infinite compact dense-in-itself metric space (X,d) can be written as an well ordered union of compact nowhere dense sets.*

14. $(\mathcal{W})$: *Every infinite compact metric space has an infinite well orderable subset.*

Before we proceed let us list a couple of results which we are going to use in the sequel.

Lemma 1.1 *[8] (i) CMSC: Compact metric spaces are second countable iff CMWOB: Compact metric spaces have a well ordered base.*
(ii) CMWOB (and consequently CMSC) implies CAC_{fin}.

A separable compact metric space (X,d) has size $\leq 2^{\aleph_0}$. (If $G = \{g_n : n \in \omega\}$ is a dense subset of X, then the mapping $x \to V_x = \{V \in \mathcal{V} : x \in V\}$, where $\mathcal{V} = \{D(g_n, 1/m) = \{x \in X : d(g_n, x) < 1/m\} : m \in \omega^+ = \omega\backslash 1, n \in \omega\}$ is 1:1 and $|\mathcal{V}| = \omega$. Hence, $|X| \leq |2^\omega|$). Since in permutation models 2^ω can be well ordered (see [5]), it follows that in such models, the statement

- CMS : *Compact metric spaces are separable*

is equivalent to the assertion

- CMWO : *Compact metric spaces are well orderable.*

Clearly CMS and CMWO imply $(\mathcal{W})$. We show in theorem 3.6 that $(\mathcal{W})$ holds in the model $\mathcal{N}$ described in theorem 3.5 but CMWO fails in $\mathcal{N}$. Thus, $(\mathcal{W})$ does not imply back CMS. In general, CMWO implies CMS (see the proof of theorem 2.1 (ii)$\to$(iii)) but the converse fails. Indeed, in Feferman's model (see [3] and model $\mathcal{M}2$ of [5]) CAC, hence CMS (see [4] and [8]), is true and the unit interval $[0, 1]$ is a separable compact metric space which is not well orderable. Thus, CMWO fails in $\mathcal{M}2$. We show in theorem 2.1 that the following weaker version of CMWO

- CMWOD : *Compact metric spaces have a well ordered dense subset*

is equivalent to each one of the statements: CMS and

- CML : *Compact metric spaces are Loeb.*

In Section 2 we also study the statement

- CMWL : *Compact metric spaces are weakly Loeb.*

In Theorems 3.5 and 3.6 we shall be concerned with the questions:

1. Is CMWL provable in ZF^0?
2. Does Form 9 imply Form 154?
3. Does CAC_{fin} imply Form 154?
4. Does Form 9 imply Form 131?

The status of 2., 3. and 4. is indicated as "unknown" in table 1 of [5]. We shall prove that the answer to all of them is in the negative.

We would like to point out here that being a metric space Loeb (weakly Loeb) is actually equivalent to AC (MC). Indeed, any set A with the discrete metric is a space in which every subset X of A is closed. Therefore the powerset $\mathcal{P}(A)$ of A has a choice function (a multiple choice function) meaning that A is well orderable (A can be written as a well ordered union of finite sets). The conclusion now follows from the well known results:
(A) [13] *AC iff every set can be well ordered*, and
(B) **Levy's Lemma** [10], *MC iff every set can be written as a well ordered union of finite sets*
respectively.

2 Main results

We begin with a list of characterizations of CML.

Theorem 2.1 *The following are equivalent: (i) CML.*
(ii) CMWOD.
(iii) CMS.
(iv) CMSC.

Proof. (i) $\rightarrow$ (ii). Fix (X, d) a metric space and let f be a choice function of the family G of all closed subsets of X. We construct recursively a dense set $D = \{d_n : n \in \aleph\}$ such that for all $m \in \aleph, B_m = \{d_n : n < m\}$ is not dense in X. For $n = 0$ put $d_0 = f(X)$. For $n = v + 1$, a non limit ordinal, and for $k \in \omega^+$, let $U_k = \cup\{D(d_j, 1/k) : j \leq v\}$. If each $U_k = X$, then B_n is a well ordered dense subset of X and the induction terminates. (If B_n is not dense then there exists an open disk $D(x, r), r > 0$ avoiding B_n. Thus, $d(d_j, x) > r/2$ for all $j \leq v$ and consequently $U_{r/2} \neq X$).

Let k_n be the first k with $U_k \neq X$ and put $d_n = f(X \backslash U_{k_n})$.

For n a limit ordinal we define U_k as in the non limit case. Again, if each $U_k = X$, then B_n is a well ordered dense subset of X and the induction terminates. Otherwise we let k_n be the first k with $U_k \neq X$ and put $d_n = f(X \backslash U_{k_n})$.

The induction surely terminates at some ordinal stage finishing the proof of (i)$\rightarrow$(ii).

(ii)$\rightarrow$(iii). Fix (X, ρ) a compact metric space and let $\mathcal{G} = \{g_i : i \in \aleph\}$ be a well ordered dense set of X. Then the collection
$\mathcal{B} = \{D(g_i, 1/n) : i \in \aleph, n \in \omega^+\}$ forms a well ordered base for the metric topology T_ρ on X. Now following the proof of Lemma 1.1(i) (this paper) given in [8] we can find a countable subcollection $\mathcal{C}$ of $\mathcal{B}$ which is a base for T_ρ. As each element of $\mathcal{C}$ is a disc we can pick its center and thus obtain a countable dense set of X.

(iii)$\rightarrow$(iv). This is straightforward.

(iv)$\rightarrow$(i). Fix (X, d) a compact metric space and let $B = \{B_n : n \in \omega\}$ be a base for X. Here is how we choose from closed sets. Fix G a closed set. Via a straightforward induction choose a nested sequence $Q_G = \{q_n : n \in \omega\}$ from B such that:
(A) $\overline{q}_{n+1} \subset q_n$ for all $n \in \omega$ and $\lim_n diameter(q_n) = 0$ and,
(B) each member of Q_G meets G non trivially.

It is easily seen that $\cap\{\overline{q}_n \cap G : n \in \omega\}$ is a singleton of G. For every closed set G, choose the unique element of $\cap\{\overline{q}_n \cap G : n \in \omega\}$. ▮

Corollary 2.2 *(i) CML implies ($\mathcal{W}$).*
(ii) Compact separable metric spaces are examples of Loeb spaces.
(iii) ($\mathcal{W}$) implies CAC_{fin}.

Proof. (i), (ii). These follow immediately from theorem 2.1.

(iii). Fix a disjoint family $\mathcal{A} = \{A_n : n \in \omega\}$ of non empty finite sets. Let $X = \bigcup \mathcal{A} \cup \{\infty\}, \infty \notin \bigcup \mathcal{A}$. Define a function $d : X \times X \to \mathbb{R}$ by requiring:

$$d(x,y) = d(y,x) = \begin{cases} 0 \text{ if } x = y \\ \max\{\frac{1}{n+1}, \frac{1}{m+1}\} \text{ if } x \in A_n, y \in A_m \text{ and } x \neq y \\ \frac{1}{n+1} \text{ if } x \in A_n \text{ and } y = \infty. \end{cases}.$$

Clearly d is a metric on X and (X,d) is a compact metric space. Let $W = \{w_i : i \in \mu\}$, μ is an infinite ordinal number, be the well ordered subset of X guarranteed by ($\mathcal{W}$). Via a straightforward induction we can define an infinite subfamily $\mathcal{B}$ of $\mathcal{A}$ having a choice set. Since CAC_{fin} is equivalent (see [14]) to the choice form PCAC_{fin}: *Every countable disjoint family of non-empty finite sets has an infinite subfamily with a choice set,* it follows that ($\mathcal{W}$) implies CAC_{fin} as required. ▮

Remark. We would like to point out here that CMS is equivalent to the statement CDMS (: Compact dense-in-itself metric spaces are separable). Clearly CMS implies CDMS. To see the converse fix a compact metric space (X,d) and let $Y = X \times 2^\omega$ be the Tychonoff product of the spaces (X,d) and $(2^\omega, \rho)$, where ρ is the metric on 2^ω defined by $\rho(f,g) = \sum_{i\in\omega} \frac{|f(i)-g(i)|}{2^{i+1}}$ for all $f, g \in 2^\omega$. Then Y is a compact dense-in-itself metrizable space (2^ω is compact and this without use of any choice principle, see [2]). Thus, Y has a countable dense subset $G = \{g_n = (p_n, q_n) : n \in \omega\}$. It is easy to see that $P = \{p_n : n \in \omega\}$ is a countable dense subset of (X,d) as required.

Next we find another characterization of CMS. A compact metric space (X,d) is separable in case it satisfies the conclusion of ($\mathcal{D}$) and conversely. Even though it is easier to check directly separability than ($\mathcal{D}$), we include this characterization here in order to stress out the resemblance and the differences between CML and CMWL respectively.

Theorem 2.3 *($\mathcal{D}$) can be added to the list of theorem 2.1.*

Proof. CMS $\to$ $(\mathcal{D})$. This is straightforward.

CMS $\leftarrow$ $(\mathcal{D})$. It suffices, in view of the latter remark, to show that $(\mathcal{D})$ $\to$ CDMS. Fix a compact dense-in-itself metric space (X, d) and let G be the dense set of the statement $(\mathcal{D})$. We describe how to choose an element from a closed proper subset A of X. We consider two cases:

(A). $A \cap G \neq \emptyset$. Let n_0 be the least integer n such that $A \cap G_n \neq \emptyset$. Suppose that $A \cap G_{n_0} = \{g_1, ..., g_k\}$. Put $r = \min\{d(g_i, g_j) : 1 \leq i, j \leq k, i \neq j\}$ and let n_1 be the least integer n such that G_n intersects only one open disk $D(g_i, r/3)$. (Notice that for each $i \leq k$ since $D(g_i, r/3)\backslash\{g_i\}$ is an open non-empty (actually infinite) set, there is a $G_n, n \neq n_0$, such that G_n meets $D(g_i, r/3)$. Now suppose that for each $n \neq n_0$ G_n meets more than one disk $D(g_i, r/3)$. Since $diameter(G_n) \to 0$, there is a $G_n, n \neq n_0$ with $diameter(G_n) < r/3$. Without loss of generality assume that G_n meets $D(g_1, r/3), D(g_2, r/3)$ in x, y respectively. Then we have that $d(g_1, g_2) \leq d(g_1, x) + d(x, y) + d(y, g_2) < 3\frac{r}{3} = r$, a contradiction). Then we may choose the center of this disk.

(B). $A \cap G = \emptyset$. We will construct inductively a nested sequence $\{A_n : n \in \omega\}$ of non empty closed subsets of A, a sequence $(r_n)_{n\in\omega}$ of positive real numbers converging to 0 and a sequence $\mathcal{Q} = \{Q_n : n \in \omega\}$ such that:

$\forall n \in \omega, \forall q, g \in Q_n, r_n = \inf\{d(q, x) : x \in A_n\} = \inf\{d(g, x) : x \in A_n\} < r_{n-1}/2$

$A_{n+1} = \cup\{B_q = \{a \in A_n : d(a, q) = r_{n+1}\} : q \in Q_n\}$.

For $n = 0$ we let $A_0 = A, r_0 = d(A, G_0)$ and $Q_0 = \{g \in G_0 : d(A, g) = r_0\}$.

Assume that $\{A_n : n \leq k\}, \{r_n : n \leq k\}$ and $\{Q_n : n \leq k\}$ have been constructed.

For $n = k + 1$ let P be the first element of $\{G_n : n \in \omega\}$ with

$0 < r_n = d(A_k, P) < r_k/2$ and $diameter(P) < r_k$.

Such a P exists because G is dense in X and $\lim_n diameter(G_n) = 0$. (Assume on the contrary that for all $n \in \omega$, $d(A_k, G_n) \geq r_k/2$. Let $x \in A_k$ and $D(x, r_k/2)$ be the open ball of radius $r_k/2$ centered at x. Then $D(x, r_k/2) \cap G = \emptyset$, contradicting the fact that G is dense. Now let $K = \{P \in \{G_n : n \in \omega\} : d(A_k, P) < r_k/2\}$. K is infinite. Otherwise, let $K = \{G_{n_1}, G_{n_2}, ..., G_{n_k}\}$ and $r = \min\{d(A_k, G_{n_i}) : 1 \leq i \leq k\}$. Then if $x \in A_k$, $D(x, r) \cap G = \emptyset$, a contradiction. Since $diameter(G_n) \to 0$, we have that $(diameter(P))_{P\in K} \to 0$. Therefore there exists $P \in K$ such that $diameter(P) < r_k$). Put

$Q_n = \{p \in P : d(A_k, p) = r_n\}$ and let $A_n = \cup\{B_q = \{a \in A_k : d(a, q) = r_n\} : q \in Q_n\}$. Since A_k is closed (thus compact) it follows that A_n being a finite union of non empty closed sets, is a non empty closed subset of X, terminating the induction.

Claim. For every $n \in \omega^+, diameter(A_n) < 2r_{n-1}$.

Proof of the claim. Fix $x, y \in A_n$. From the construction of A_n it follows that there exist $p, q \in Q_n$ such that $x \in B(p, r_n)$ (the closed ball of radius r_n centered at p) and $y \in B(q, r_n)$. We have: $d(x,y) \leq d(x,p)+d(p,q)+d(q,y) < (r_{n-1}/2) + r_{n-1} + (r_{n-1}/2) = 2r_{n-1}$ as required.

As $\delta_n = diameter(A_n) \rightarrow 0$, it follows by the compactness of X that $L_A = \cap\{A_n : n \in \omega^+\}$ is a singleton of A. Choose from A this unique element of L_A. ∎

Theorem 2.4 *(i) CMWL implies CMWS.*
(ii) CMWL implies every compact metric space has a base which can be written as a countable union of finite sets.
(iii) CMWL+CAC$_{fin}$ iff CMSC.

Proof. (i). This is a straightforward consequence of theorem 2.1.

(ii). Fix (X, d) a compact metric space, then, by CMWL, let $\mathcal{D} = \cup\{D_i : i \in \aleph\}$, $0 < |D_i| < \omega$, be dense in X. For each $n \in \omega^+$ define $\mathcal{U}_{1/n} = \{O_i = \cup\{D(x, 1/n) : x \in D_i\} : i \in \aleph\}$. Clearly $\mathcal{U}_{1/n}$ is an open cover of the compact space X. Let $m_n = \min\{m \in \omega : \exists\mathcal{V}, \mathcal{V}$ is a subcover of $\mathcal{U}_{1/n}$ of size $m\}$. Since $[\aleph]^{m_n}$ (the set of all subsets of $\aleph$ having size m_n) is well ordered and
$S = \{A \in [\aleph]^{m_n} : \{O_i : i \in A\}$ is a subcover of $\mathcal{U}_{1/n}\} \neq \emptyset$,
we may put
$A_{m_n} = \min(S)$ and $\mathcal{V}_{1/n} = \{D(x, 1/n) : x \in D_i,\, i \in A_{m_n}\}$.
Clearly $\mathcal{V}_{1/n}$ is a finite set and $\mathcal{V} = \cup\{\mathcal{V}_{1/n} : n \in \omega^+\}$ is a base for the metric topology T_d.

(iii). This follows from (ii) and the fact that CAC$_{fin}$ implies *the union of countably many finite sets is countable* (see [5]). ∎

In [7], lemma 2, it has been established that: *CMC iff every compact pseudometric space (X, d) has a dense subspace Y which is written as a countable union of finite sets.* As the proof of $(\rightarrow)$ goes through with metric in place of pseudometric, it follows that CMC implies: *Every compact metric space (X, d) has a dense subspace Y which is written as a countable union of finite sets.*
Thus, CMC implies CMWS.

Theorem 2.5 *Form 154 implies statement* $(\mathcal{W})$.

Proof. Fix (X,d) an infinite compact metric space. Clearly the Tychonoff product X^ω is metrizable and by 154 X^ω is compact. Since X is an infinite compact space it has at least one accumulation point x. Let $\mathcal{V}_x = \{V_n : n \in \omega\}$ be an open neighborhood base of x. We construct by induction a strictly decreasing subfamily $\{V_{n_i} : i \in \omega\}$ of $\mathcal{V}_x$ such that

$\forall i \in \omega$, $\overline{V_{n_i}} \backslash V_{n_{i+1}} \neq \emptyset$.

For $i = 0$ put $V_{n_0} = V_0$.

Assume that V_{n_i}, $i \in k+1$, have been constructed. Let n_x be the least integer n such that $D(x, 1/n) \subseteq V_{n_k}$. Since x is an accumulation point, there is an element $y \in X$ such that $y \in D(x, 1/n_x)\backslash\{x\}$. Put $r = \min\{\frac{1}{n_x+1}, d(x,y)\}$ and let n_{k+1} be the least n such that $\overline{V_n} \subseteq D(x,r)$. Clearly $\overline{V_{n_k}} \backslash V_{n_{k+1}} \neq \emptyset$. The induction terminates.

For each $i \in \omega$, define $A_i = \prod_{j \in i+1}(\overline{V_{n_j}} \backslash V_{n_{j+1}}) \times X^{\omega \backslash i+1}$. Now $A = \{A_i : i \in \omega\}$ is a descending family of non empty closed sets in X^ω, thus by the compactness of X^ω, $\cap A \neq \emptyset$. It is straightforward to verify that any $x \in \cap A$ yields a countably infinite subset of X. ∎

3 Models

Theorem 3.1 *(i) CML is not provable in* ZF^0.
(ii) CMC and statement P, $P \in \{$*CML*, $(\mathcal{D})$, $(\mathcal{W})\}$, *are mutually independent.*
(iii) CMWL does not imply either of CML, CMC.
(iv) $Q \in \{$*CML, CMWO*, $(\mathcal{D})$, $(\mathcal{W})\}$ *does not imply form 9.*
(v) $(\mathcal{W})$ *does not imply either of CAC, CMWO.*

Proof. (i), (ii), (iii). In the second Fraenkel model $\mathcal{N}2$ in [5] MC, and consequently CMC and CMWL, is true whereas CAC_{fin} is false. Therefore by theorem 2.1 and corollary 2.2 each statement P fails in $\mathcal{N}2$. Now in the basic Fraenkel model $\mathcal{N}1$ in [5] CMSC is true (theorem 3.2 in [8]) and consequently by theorem 2.1 and corollary 2.2 each statement P and CMWL (CML clearly implies CMWL) are true in $\mathcal{N}1$. However CMC fails in $\mathcal{N}1$ meaning that P and CMWL do not imply CMC.

(iv). Form 9 fails in the basic Fraenkel model (see [5]) while Q is true.

(v). In the basic Cohen model $\mathcal{M}1$ in [5] form 154 is true (see [5]), therefore by theorem 2.5, $(\mathcal{W})$ is true in $\mathcal{M}1$. On the other hand CAC is false in $\mathcal{M}1$ and $[0,1]$ is a compact metric space which is not well orderable in $\mathcal{M}1$. ∎

Theorem 3.2 *(i)* $(\oplus)$ *implies Form 79 in [5]:* $\mathbb{R}$ *can be well ordered.*
(ii) $(\pm)$ *implies Form 170 in [5]:* $\aleph_1 \leq |\mathbb{R}|$.

Proof. (i). The unit interval $[0,1]$ with the usual metric is a compact metric space. Thus, by $(\oplus)$ $[0,1]$ can be expressed as a well ordered union of finite sets, say $\cup\{F_i : i \in \aleph\}$. Then as each F_i with the usual ordering is well ordered it follows that $[0,1]$, hence $\mathbb{R}$, being the well ordered union of well ordered sets is well ordered.

(ii). Let $[0,1] = \cup\{F_i : i \in \aleph\}$ with each F_i a closed nowhere dense subset of $[0,1]$. As $[0,1]$ is a second countable metric space, it follows that $\aleph_1 \leq \aleph$. Furthermore, as the family of all closed sets of $\mathbb{R}$ has size $2^{\aleph_0}$, it follows that $\mathbb{R}$ has a subset of size $\aleph_1$ finishing the proof of the theorem. ∎

Lemma 3.3 *CMWL does not imply* $(\pm)$*, hence also* $(\oplus)$*, in ZF.*

Proof. In Shelah's Model II, see [15] and model $\mathcal{M}38$ of [5], CAC, hence CMWL holds but 170 fails. Thus, $(\pm)$ fails also. ∎

Strangely enough CMWL implies $(\pm)$ in all permutation models.

Theorem 3.4 *In all permutation models CMWS implies* $(\pm)$*.*

Proof. Let $\mathcal{N}$ be a permutation model and let (X,d) be a compact dense-in-itself metric space in $\mathcal{N}$. By CMWS X has a dense subset $G = \cup\{G_i : i \in \omega\}$, $|G_i| < \omega$. Define a relation $\sim$ on X by requiring:

for all $x, y \in X$, $x \sim y$ iff $(\forall n \in \omega)\ d(x, G_n) = d(y, G_n)$.

It is straightforward to verify that $\sim$ is an equivalence relation on X.

Claim 1. For each $x \in X$, the equivalence class $[x]$ of x is a closed set.

Proof of claim 1. Fix an element $y \in X \backslash [x]$. Then $d(x, G_n) \neq d(y, G_n)$ for some $n \in \omega$. There are two cases:
(i) $d(x, G_n) > d(y, G_n)$. Then $r = d(x, G_n) - d(y, G_n) > 0$ and $D(y, r/2) \cap [x] = \emptyset$. Otherwise let $z \in D(y, r/2) \cap [x]$ and $t \in G_n$ such that $d(y, G_n) = d(y, t)$. Then $d(x, G_n) = d(z, G_n) \leq d(z, t) \leq d(z, y) + d(y, t) < \frac{d(x,G_n)}{2} + \frac{d(y,G_n)}{2} < \frac{d(x,G_n)}{2} + \frac{d(x,G_n)}{2} = d(x, G_n)$. A contradiction.

(ii) $d(x, G_n) < d(y, G_n)$. Let $r = d(x, G_n)$ and $U = \{z \in X : d(z, G_n) \leq r\}$. Clearly U is a closed set such that $[x] \subseteq U$ and $y \notin U$.

Thus, $[x]$ is a closed set finishing the proof of claim 1. ▮ (claim 1)

Claim 2. $[x]$ is a nowhere dense set.

Proof of claim 2. Fix $n \in \omega$. For each $g \in G_n$ define $Y_g = \{y \in [x] : d(y, g) = r\}$, $r = d(x, G_n)$. Since the metric d is a continuous function we see that Y_g is closed. Further, Y_g is nowhere dense and $[x] = \cup\{Y_g : g \in G_n\}$ (Assume the contrary and let U be a non-empty open set such that $U \subseteq Y_g$. Since X is dense-in-itself, U is infinite. Fix an $n^* \in \omega$ and elements $y \in U \cap G_{n^*}$ and $z \in U \backslash G_{n^*}$. Then $d(y, G_{n^*}) = 0 < d(z, G_{n^*})$ meaning that $y \not\sim z$. This contradicts the fact that $y, z \in [x]$). Since a finite union of nowhere dense sets is a nowhere dense set we conclude that $[x]$ is also nowhere dense. This completes the proof of claim 2. ▮ (claim 2)

Claim 3. $|X/\sim| \leq |\mathcal{P}(\mathbb{R})|$, where $X/\sim$ is the set of all equivalence classes.

Proof of Claim 3. Define a function $f : X/\sim \to \mathcal{P}(\mathbb{R} \times \omega)$ by requiring for all $x \in X$, $f([x]) = \{(r, n) : d(x, G_n) = r\}$. f is one-to-one (if $x \not\sim y$, then there is an $n \in \omega$ such that $d(x, G_n) \neq d(y, G_n)$. Thus, $f([x]) \neq f([y])$). ▮ (claim 3)

Since in all permutation models the powerset of $\mathbb{R}$ can be well ordered, we conclude by claim 3 that $X/\sim$ can be well ordered. This finishes the proof of the theorem. ▮

Theorem 3.5 *(i) CMWL is not provable in ZF^0.*
(ii) Form 9 does not imply CMWL.
(iii) CAC_{fin} does not imply either of CMS, CMWS.

Proof. (i) We first give the description of a permutation model $\mathcal{N}$. The set of atoms $A = \cup\{A_n = \{a_{nt} = (1/n)(\cos t, \sin t) : t \in [0, 2\pi)\} : n \in \omega^+\}$. The group of permutations $\mathcal{G}$ is the group of all permutations on A which rotate the A_n's by an angle θ_n and supports are finite. It has been shown in [9] that the family $\{A_n : n \in \omega^+\}$ does not have a multiple choice function in $\mathcal{N}$ (in fact, $\{A_n : n \in \omega^+\}$ does not have a choice function in $\mathcal{N}$).

Adjoin a point $*$ to A and define a function $d^* : A(*) \times A(*) \to \mathbb{R}$ by requiring:

$$d^*(x,y)=d^*(y,x)=\begin{cases}\max\{1/n,1/m\} \text{ if } x\in A_n, y\in A_m \text{ and } n\neq m\\ \rho(x,y) \text{ if } x,y\in A_n\\ 1/n \text{ if } x\in A_n \text{ and } y=*\\ 0 \text{ if } x=y=*\end{cases},$$

where ρ is the Euclidean metric. It can be readily verified that $(A(*), d^*)$ is a compact metric space in $\mathcal{N}$ (each (A_n, ρ) is a compact metric space, see also [9]). We claim that $(A(*), d^*)$ is not weakly Loeb. Assume the contrary and let $G = \cup\{G_n : n \in \omega\}, |G_n| < \omega$, be the dense set of $A(*)$ which is guarandeed by theorem 2.4. Since each A_m is an open set, it follows that A_m meets a G_n (actually infinitely many). Let m_n be the first $n \in \omega$ such that $A_m \cap G_n \neq \emptyset$. Then $\{A_m \cap G_{m_n} : m \in \omega^+\}$ is a multiple choice set of the family $\{A_n : n \in \omega^+\}$ which is a contradiction. Thus, CMWL fails in $\mathcal{N}$.

(ii) Let $\mathcal{N}$ be the permutation model defined in (i) and let X be any infinite set in $\mathcal{N}$. If X is well orderable then there is nothing to show. Assume that X is not well orderable and let E be a support of X. Fix $a \in X$ such that E is not a support of a. For each $n \in \omega^+$ and $t \in [-\pi, \pi]$ let $g_{nt} \in \mathcal{G}$ be the permutation of A which fixes all $A_m, m \in \omega^+, m \neq n$ and rotates A_n by t i.e. $g_{nt}(a_{my}) = \begin{cases} a_{my} \text{ if } n\neq m\\ (1/n)(\cos(y+t), \sin(y+t)) \text{ if } n=m\end{cases}$.
Fix $n \in \omega^+$ and $t \in [-\pi, \pi]$ such that $g_{nt} \in fix(E) = \{\pi : \pi(E) = E$ pointwise$\}$ and $g_{nt}(a) \neq a$. Note that such a pair (n, t) exists because otherwise E would be a support of a. Let F be a support of a and $D = \{g_{ns}(a) : s \in [-\pi, \pi]\}$. It can be readily verified that F is a support for D. Hence $D \in \mathcal{N}$. Moreover $D \subseteq X$ as $g_{ns} \in fix(E)$ and since $[-\pi, \pi]$ is well orderable in $\mathcal{N}$, it follows that D is well orderable also. We will be done once we show:

Claim. D is infinite.

Proof of the claim. Assume the contrary and let $H = \{t \in [-\pi, \pi] : g_{nt}(a) = a\}$. H is a subgroup of the group $Q = ([-\pi, \pi], \oplus)$ where $\oplus$ is addition modulo 2π.

Q/H is infinite. If not then Q/H has finite order, say $k > 1$. Fix an element $t \in (Q \backslash H)$. Without loss of generality we may assume that $0 < t \leq \pi$. Then, $t/k \leq \pi/k$ and $k(t/k) = t/k \oplus t/k \oplus, ..., \oplus t/k = t/k + t/k +, ..., +t/k = t$. It follows that $H = ((t/k) \otimes H)^k = k(t/k) \otimes H = t \otimes H$, where $\otimes$ denotes the addition operation on Q/H. This is a contradiction because $t \notin H$. Thus, Q/H is infinite as required.

Let t, s belong to different cosets of Q/H. If $g_{nt}(a) = g_{ns}(a)$, then $g_{n(t-s)}(a) = a$ meaning that $t - s \in H$ which is a contradiction. Thus, D is an infinite subset of X finishing the proof of the claim and of (ii).

(iii). Since form 9 implies CAC_{fin}, the latter holds in the model $\mathcal{N}$ defined in (i). Now the proof of (i) applies to show that CMS and CMWS fail in $\mathcal{N}$.∎

Theorem 3.6 *Let $\mathcal{N}$ be the permutation model defined in the proof of theorem 3.5. Then the following hold:*
(i) $(\oplus)$, $(\mathcal{D})$ fail in $\mathcal{N}$.
(ii) Each statement $P \in \{154, 343\}$ fails in $\mathcal{N}$.
(iii) $(\pm)$ fails in $\mathcal{N}$.
(iv) $(\mathcal{W})$ holds in $\mathcal{N}$.
(v) AC(WO, WO) (the axiom of choice for well ordered families of well orderable sets) fails in $\mathcal{N}$.
(vi) Form 131 fails in $\mathcal{N}$.

Proof. (i). The space $(A(*), d^*)$ defined in the proof of theorem 3.5(i) is a compact dense-in-itself metric space in $\mathcal{N}$ which satisfies none of $(\oplus)$, $(\mathcal{D})$.

(ii). By theorem 3.5 we get that $\{(A_n, \rho) : n \in \omega^+\}$ is a family of compact metric spaces with empty product. Thus, 343 fails in $\mathcal{N}$. Moreover, 154 is false in $\mathcal{N}$ as otherwise $\prod_{n\in\omega^+} A_n$ would be non empty.

(iii). Assume on the contrary that $(\pm)$ is true in $\mathcal{N}$. Then for the compact dense-in-itself metric space $(A(*), d^*)$ there exists a well ordered family of compact nowhere dense sets $\{W_i : i \in \aleph\}$ such that $A(*) = \cup\{W_i : i \in \aleph\}$. For each $n \in \omega^+$, let i_n be the least $i \in \aleph$ such that $A_n \cap W_i \neq \emptyset$. Clearly $A_n \cap W_{i_n}$ is a nowhere dense set and since A_n is open, $A_n \cap W_{i_n} \subset A_n$ (strict inclusion). To complete the proof it suffices to show that $\mathcal{A} = \{A_n \cap W_{i_n} : n \in \omega^+\} \notin \mathcal{N}$. Assume the contrary and let E be a support for $\mathcal{A}$ and $n_0 = \max\{n \in \omega^+ : E \cap A_n \neq \emptyset\}$ (E being a finite set intersects finitely many A_n's). Fix $m > n_0$ and $x, y \in A_m \cap W_{i_m}$, $x \neq y$. Without loss of generality we may assume that y is the image of x under a rotation by positive angle. Let z in the arc $\widehat{xy}$, $z \neq x, y$ and $r = \min\{\rho(x,z), \rho(y,z), \frac{1}{m(m+1)}, \frac{1}{m(m-1)}\}$. Then the open disk $D(z,r) \nsubseteq A_m \cap W_{i_m}$. Let $w \in D(z,r) \backslash (A_m \cap W_{i_m})$ and ϑ be the central angle corresponding to the arc $\widehat{xw}$. Consider the permutation π on A which is the identity map on each $A_n, n \neq m$ and rotates A_m by θ. Clearly $\pi \in fix(E)$ so $\pi(\mathcal{A}) = \mathcal{A}$. But $\pi(A_m \cap W_{i_m}) \neq A_m \cap W_{i_m}$. This contradiction completes the proof of (iii).

(iv). Since form 9 clearly implies $(\mathcal{W})$ and form 9 holds in $\mathcal{N}$ we have that $(\mathcal{W})$ is also true in $\mathcal{N}$.

(v). Each subset A_n of the atoms is a well orderable set (if $a \in A_n$, then $fix(\{a\}) \subseteq fix(A_n)$) and $\{A_n : n \in \omega^+\}$ is a countable family in $\mathcal{N}$ having no choice set.

(vi). Assume on the contrary that form 131 holds in $\mathcal{N}$. Then for the family $\mathcal{A} = \{A_n : n \in \omega^+\}$ (which is $\mathcal{N}$ since it has empty support) there exists a function $f \in \mathcal{N}$ such that $f(A_n)$ is a non-empty countable subset of A_n. Let E be a support for f and $n_0 = \max\{n \in \omega^+ : E \cap A_n \neq \emptyset\}$. Fix $m > n_0$ and $x, y \in f(A_m)$. Since $f(A_m)$ is a countable set and the arc $\widehat{xy}$ has size $|\mathbb{R}|$, there exists a $z \in \widehat{xy} \backslash f(A_m)$. Let ϑ be the central angle corresponding to the arc $\widehat{xz}$. Consider the permutation π on A which is the identity map on each $A_n, n \neq m$ and rotates A_m by θ. Clearly $\pi \in fix(E)$ so $\pi(f) = f$. But $\pi(f(A_m)) \neq f(A_m)$ meaning that f is not a function. This contradiction completes the proof of (vi) and of the theorem. ∎

4 Summary

The following diagram presents the results of the paper.

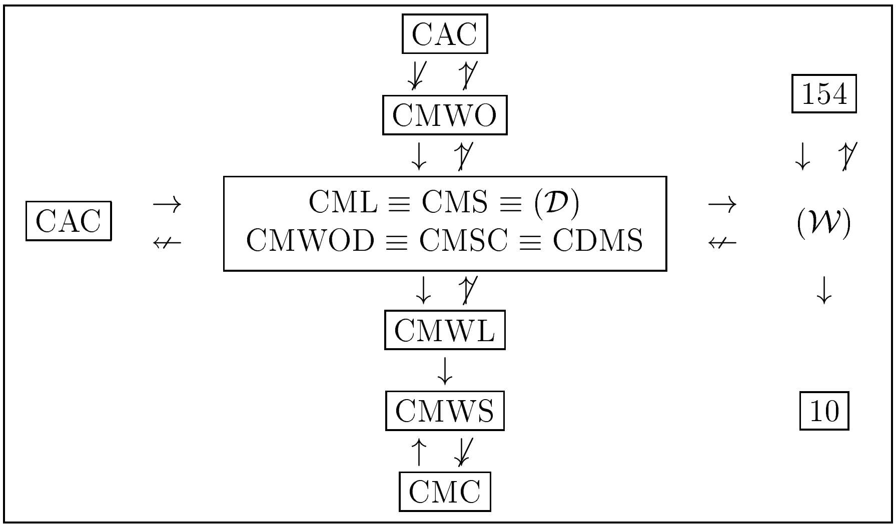

*Diagram*1

The independence results are

1. (a) CMWL is not provable in ZF^0 (Theorem 3.5). Thus, from diagram1 we have that none of the statements CMWO, CML and its equivalents listed in diagram1 are provable in ZF^0.
   (b) CMWL does not imply either of CML, CMC in ZF^0 (Theorem 3.1).
   (c) CMWL does not imply either of $(\pm)$, $(\oplus)$ in ZF (Lemma 3.3).

2. CMWS does not imply CMC (Theorem 2.4 and Theorem 3.1).

3. CAC and CMWO are mutually independent statements (Introduction and Theorem 3.1).

4. CMS and its equivalents listed in diagram1 do not imply CMWO in ZF (Introduction).

5. (a) $Q \in \{\text{CML, CMWO}, (\mathcal{D}), (\mathcal{W})\}$ does not imply form 9 (Theorem 3.1).
   (b) Form 9 does not imply the statements CMWL, 154 and 131 (Theorem 3.5 and Theorem 3.6 respectively). Therefore, from diagram1 we have that form 9 implies none of the statements CMWO, CML and its equivalents listed in diagram1.

6. Form 10 implies none of the statements CMWS, CMS and 154 (Theorem 3.5, Theorem 3.6).

7. CMC and statement P, $P \in \{\text{CML}, (\mathcal{D}), (\mathcal{W})\}$, are mutually independent (Theorem 3.1).

8. $(\mathcal{W})$ implies none of the statements CAC, CMWO and 154 (Theorem 3.1, Theorem 3.6).

9. The statements $(\oplus)$, $(\pm)$ are not provable in ZF^0 (Theorem 3.6).

Questions. (i) Does any of CMWS, CMC imply CMWL?
(ii) Does (10) imply $(\mathcal{W})$?

References

[1] N. Brunner, *Products of compact spaces in the least permutation model*, Zeitschr. f. math. Logic und Grundlagen d. Math. **31** 1985.

[2] N. Brunner, *Amorphe Potenzen kompakter Räume,* Arch. Math. Logik Grundlagenforsch **24**, 1984.

[3] S. Feferman, *Applications of forcing and generic sets*, Fund. Math. **56** 1965.

[4] C. Good and I. J. Tree, *Continuing horrors of topology without choice*, Topol. Appl. **63** 1995.

[5] P. Howard and J. E. Rubin, *Consequences of the axiom of choice,* Math. Surveys and Monographs, **59** A.M.S. Providence R.I., 1998.

[6] K. Keremedis, *Disasters in topology without the axiom of choice,* preprint 1997.

[7] K. Keremedis, *On Weierstrass compact pseudometric spaces and a weak form of the axiom of choice,* accepted in Topol. Appl.

[8] K. Keremedis and E. Tachtsis, *On Lindelöf metric spaces and weak forms of the axiom of choice*, Math. Logic Quarterly, **46**, 2000.

[9] K. Keremedis and E. Tachtsis, *On Loeb and weakly Loeb Hausdorff spaces*, preprint 1999.

[10] A. Levy, *Axioms of multiple choice,* Fund. Math. **50** (1962), 475-483.

[11] P. A. Loeb, *Proof of Tychonoff's theorem*, Am. Math. Monthly **72** 1965.

[12] M. Morillon, *Topologie, Analyse Nonstandard et Axiome du Choix*, Thesis, Universite Blaise Pascal.

[13] H. Rubin and J. E. Rubin, *Equivalents of the Axiom of Choice* II, North-Holland**,** 1985.

[14] G. P. Shannon, *Equivalent versions of a weak form of the axiom of choice*, Notre Dame J. Formal Logic **29** 1983.

[15] S. Shelah, *Can you take Solovay's inaccessible away*?, Israel J. Math. **48**, 1-47