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On the Exponent of the NK_0-Groups of Virtually Infinite Cyclic Groups

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Abstract. It is known that the K-theory of a large class of groups can be computed from the K-theory of their virtually infinite cyclic subgroups. On the other hand, Nil-groups appear to be the obstacle in calculations involving the K-theory of the latter. The main difficulty in the calculation of Nil-groups is that they are infinitely generated when they do not vanish. We develop methods for computing the exponent of NK_0-groups that appear in the calculation of the K_0-groups of virtually infinite cyclic groups.

1 Introduction

The results of Farrell-Jones [11] have shown that the computation of the lower K-theory of a large class of groups depends on our ability to calculate the K-theory of virtually infinite cyclic groups, *i.e.*, groups that contain a subgroup, of finite index, isomorphic to the infinite cyclic group. Virtually infinite cyclic groups are two-ended groups and they are of two types [9, Theorem 6.12]:

- Type (1): Groups that admit an epimorphism, with kernel a finite group, to the infinite cyclic group $\mathbb{Z}$.
- Type (2): Groups that admit an epimorphism, with finite kernel, to the infinite dihedral group D_∞.

The groups of the first type have the form $\Gamma = H \rtimes_\alpha \mathbb{Z}$ with H a finite group. In this case, the twisted Bass-Heller-Swan formula can be used for the calculation of the K-theory of Γ:

$$K_0(\mathbb{Z}H) \xrightarrow{1-\alpha_*} K_0(\mathbb{Z}H) \longrightarrow K_0(\mathbb{Z}\Gamma)/\mathcal{N} \longrightarrow K_{-1}(\mathbb{Z}H) \xrightarrow{1-\alpha_*} K_{-1}(\mathbb{Z}H)$$

where $\mathcal{N} = \mathrm{NK}_0(\mathbb{Z}H, \alpha) \oplus \mathrm{NK}_0(\mathbb{Z}H, \alpha^{-1})$. The Nil-groups that appear are the twisted versions of the classical Nil-groups introduced by Bass [1], [10]. In the case that α is the identity, the exponents of the corresponding NK_0-groups were computed in [5]. We will generalize their methods to the twisted NK_0-groups.

The algebraic structure of the groups in the second class is more complicated. A group Γ in the second class has the form $\Gamma = G_0 *_H G_1$ where the groups G_i, $i = 0, 1$, and H are finite and $[G_i : H] = 2$. The K_0-group of $\mathbb{Z}\Gamma$ fit into an exact sequence:

$$\begin{aligned} K_0(\mathbb{Z}H) \to K_0(\mathbb{Z}G_0) \oplus K_0(\mathbb{Z}G_1) \to K_0(\mathbb{Z}\Gamma)/\mathcal{N} \\ \to K_{-1}(\mathbb{Z}H) \to K_{-1}(\mathbb{Z}G_0) \oplus K_{-1}(\mathbb{Z}G_1). \end{aligned}$$

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In this case, the Nil-group involved is Waldhausen's Nil-group [23], [24], $\mathcal{N} = \widetilde{\mathrm{Nil}}^W_{-1}(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_1 - H])$. We will use the special structure of Γ to extend the methods of [16] and [17] to this case. Again, our methods heavily depend on the methods in [5].

In both cases, the Nil-groups are direct summands of the K-groups of $\mathbb{Z}\Gamma$. Also, if the Nil-terms are ignored, both exact sequences are the analogue of Mayer-Vietoris sequences for K-groups. The last remark is consistent with the general belief that Nil-groups are the obstruction groups for K-theory to be a homology theory. On the other hand, the "homology part" of K-theory, in many instances, can be identified with an appropriate analogue of controlled K-theory [15]. Our interest in the exponent of the Nil-groups of virtually infinite cyclic groups arises from the fact that Nil-groups measure the difference between controlled and non-controlled phenomena. In many instances that difference is reflected on phenomena that are completely topological in nature [6]. In a forthcoming paper [7] we will use the results on the exponent of the Nil-groups to compute the exponent of equivariant topological K-groups.

The main result of the paper can be summarized as follows: Let $\Gamma = H \rtimes_\alpha \mathbb{Z}$ or $\Gamma = G_0 *_H G_1$ where H is a finite group of order n and, in the second case, $[G_0 : H] = [G_1 : H] = 2$. We call H the *base* group of the virtually infinite cyclic group. We outline a construction for a positive integer n. Let $n = p_1^{k_1} p_2^{k_2} \cdots p_s^{k_s}$ be the decomposition of n into prime numbers. Choose ℓ_i, $i = 1, \ldots, s$, integers such that $p_i^{\ell_i} \geq k_i n$ for all i and set $n' = p_1^{\ell_1} p_2^{\ell_2} \cdots p_s^{\ell_s}$.

***Theorem* (Main Theorem)** *The Nil-groups (twisted Bass-Heller-Swan Nil-groups in the Type (1) case, Waldhausen's Nil-groups in the Type (2) case) that appear in the splitting of the virtually infinite cyclic groups with base a finite group of order n have exponent n'.*

The method of the proof follows closely the ideas developed in [5]. But the methods really go back to [3] and [12]. The main calculational tool is that for any finite group H of order n and an automorphism $\alpha\colon H \to H$, there is a pull-back diagram

$$\begin{tikzcd} \mathbb{Z}H \arrow[r] \arrow[d] & \mathcal{M} \arrow[d] \\ \mathbb{Z}H/n\mathcal{M} \arrow[r] & \mathcal{M}/n\mathcal{M} \end{tikzcd}$$

where $\mathcal{M}$ is a hereditary order such that

(1) $\mathbb{Z}H \subset \mathcal{M} \subset \mathbf{Q}H$.
(2) $\mathcal{M}$ is α invariant.
(3) $n\mathcal{M} \subset \mathbb{Z}H$.

For the Nil-groups of Type (1) groups, the calculations of [5] can be generalized in the twisted setting using the α-invariant order $\mathcal{M}$. In the Type (2) case, we use the description of the lower Waldhausen's Nil-groups as the reduced K-groups of an augmented ring [6], [17], [16]. The special structure of the groups in Type (2) allows us to express the augmented ring as a subring of the ring of the 2×2-matrices over a twisted polynomial ring. Methods similar to the ones in [5] can be applied to complete the proof in this case.

In the last section, we use use the main result to compute the $\tilde{\mathrm{K}}_0$-groups of certain groups of Type (2) with base group cyclic of order a power of 2. The computation is done after inverting the prime 2. We would like to remark that the methods and the basic ideas used in Type (2) groups were developed in [17]. In [17], all the calculations are based on the fact that all the rings appearing are commutative. In the case of virtually infinite cyclic groups of Type (2), the underlying rings are not commutative but the bimodules are “almost” isomorphic to the basic ring (only the right action is twisted by an automorphism). That is the reason that the methods in [17] work also in this case.

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2 Twisted Nil-Groups

This section is basically a modification of the argument given in [5] on the exponent of NK_0-groups.

Let H be a finite group of order n and $\alpha\colon H \to H$ a group automorphism. We write $R = \mathbb{Z}H$. We denote by α the automorphism induced on R by α. Our objective is to calculate the exponent of the twisted Nil-group $\mathrm{NK}_0(R, \alpha)$. This group is defined as the kernel of the augmentation map

$$\varepsilon\colon K_0(R_\alpha[t]) \to K_0(R).$$

The map ε is a split epimorphism with splitting given by the inclusion induced map. Thus $\mathrm{NK}_0(R, \alpha)$ is a summand of $K_0(R_\alpha[t])$. The group $\mathrm{NK}_0(R, \alpha)$ appears in the twisted Bass-Heller-Swan formula given in [10] which calculates the K-theory of the twisted Laurent ring:

$$K_0(R) \xrightarrow{1-\alpha_*} K_0(R) \to K_0(R[t, t^{-1}])/\mathcal{N} \to K_{-1}(R) \xrightarrow{1-\alpha_*} K_{-1}(R)$$

where $\mathcal{N} = \mathrm{NK}_0(R, \alpha) \oplus \mathrm{NK}_0(R, \alpha^{-1})$. Actually the formula in [10] is given for K_1-groups only but its extension to lower K-theory can be achieved using the techniques in [20, Section 11].

Remark 2.1 In [10, Theorem 25], it is shown that if a ring R is regular and α is an automorphism of R, then the twisted polynomial and Laurent extensions of R are regular rings. In particular, in this case all the NK-groups vanish [19, p. 114].

The main tool for adjusting the proof in [5] to our case is the existence of a hereditary order $\mathcal{M}$ with the properties stated in the introduction. Another important ingredient is the twisted version of Proposition 2.12 in [1]. Recall that a ring S is called *quasi-regular* if there is a two-sided nilpotent ideal J of S such that S/J is regular. We adjust this classical definition to our setting.

Definition 2.2 Let R be a ring and α is an automorphism of R. Then R is called α-quasi-regular if there is a two-sided nilpotent ideal which is α invariant such that R/J is regular.

Remark 2.3 Let R be an Artinian ring and α an automorphism of R. Then the Jacobson radical J of R is a two-sided nilpotent ideal and it is α-invariant. Thus R is α-quasi-regular for each automorphism α.

Lemma 2.4 *Let R be a ring and $\alpha\colon R \to R$ a ring automorphism. If R is α-quasi-regular then*

$$\mathrm{NK}_i(R,\alpha) = 0, \quad \textit{for } i \leq 0.$$

Proof We will show the result for $i = 0$. The other cases follows from standard methods of lower K-theory. There is exists an α-invariant two-sided ideal J of R such that R/J is regular. Then the ring $(R/J)_\alpha[t]$ is also regular. Consider the commutative diagram induced by ring projections

$$\begin{array}{ccccc}
K_0(R_\alpha[t]) & \longrightarrow & K_0\big((R/J)_\alpha[t]\big) & \xrightarrow{\cong} & K_0(R_\alpha[t]/J_\alpha[t]) \\
\downarrow & & \downarrow & & \downarrow \\
K_0(R) & \longrightarrow & K_0(R/J) & \xrightarrow{=} & K_0(R/J).
\end{array}$$

The composite horizontal maps are isomorphisms because $J_\alpha[t]$ and J are nilpotent ideals. The middle vertical map is an isomorphism because R/J is regular (Remark 2.1). Thus the first vertical map is also an isomorphism. The result follows. ∎

We can now outline the argument given in [5]. The number n' is as in the introduction.

Theorem 2.5 *With the above notation, the group* $\mathrm{NK}_0(R,\alpha)$ *has exponent* n'.

Proof Choose an α-invariant hereditary order $\mathcal{M}$ as in the introduction. There are pull-back diagrams

$$\begin{array}{ccc}
R & \longrightarrow & \mathcal{M} \\
\downarrow & & \downarrow \\
R/n\mathcal{M} & \longrightarrow & \mathcal{M}/n\mathcal{M}
\end{array}
\qquad
\begin{array}{ccc}
R_\alpha[t] & \longrightarrow & \mathcal{M}_\alpha[t] \\
\downarrow & & \downarrow \\
(R/n\mathcal{M})_\alpha[t] & \longrightarrow & (\mathcal{M}/n\mathcal{M})_\alpha[t]
\end{array}$$

(we still denote by α the automorphism induced by α on all the rings involved) which induces a commutative diagram of exact sequences

$$\begin{array}{ccccc}
K_1\big((\mathcal{M}/n\mathcal{M})_\alpha[t]\big) & \longrightarrow & K_0(R_\alpha[t]) & \longrightarrow & K_1\big((R/n\mathcal{M})_\alpha[t]\big) \oplus K_1(\mathcal{M}_\alpha[t]) \\
\downarrow & & \downarrow & & \downarrow \\
K_1(\mathcal{M}/n\mathcal{M}) & \longrightarrow & K_0(R) & \longrightarrow & K_1(R/n\mathcal{M}) \oplus K_1(\mathcal{M}).
\end{array}$$

Since the vertical maps are split epimorphisms, taking kernels results to an exact sequence

$$\mathrm{NK}_1(\mathcal{M}/n\mathcal{M},\alpha) \xrightarrow{\partial} \mathrm{NK}_0(R,\alpha) \to \mathrm{NK}_0(R/n\mathcal{M},\alpha) \oplus \mathrm{NK}_0(\mathcal{M},\alpha).$$

Since $\mathcal{M}$ is regular, $\mathrm{NK}_0(\mathcal{M},\alpha) = 0$. Since $R/n\mathcal{M}$ is finite, it is artinian. By Remark 2.3, $R/n\mathcal{M}$ is α-quasi-regular and by Lemma 2.4, $\mathrm{NK}_0(R/n\mathcal{M},\alpha) = 0$. Thus the map ∂ is onto. So it is enough to compute the exponents of a set of generators of $\mathrm{NK}_1(\mathcal{M}/n\mathcal{M})$.

As in [5, Corollary 3.6], the map

$$1 + tJ_\alpha[t] \to \mathrm{NK}_1(\mathcal{M}/n\mathcal{M},\alpha)$$

is onto. But the elements of the group $1 + tJ_\alpha[t]$ have exponent n' [5, p. 10]. The result follows. ■

Remark 2.6 Using the arguments in [5, Proposition 3.7], the elements of the form $B = \{1 + jt^s : j \in J, n \in \mathbb{N}\}$ generate the group $1 + tJ_\alpha[t]$. There is an action of the multiplicative monoid $\mathbb{N}$ on B by $n.(1 + jt^s) = 1 + jt^{sn}$, which induces an action on $1 + tJ_\alpha[t]$. Under this action $1 + tJ_\alpha[t]$ becomes a $\mathbb{Z}[\mathbb{N}]$-module with basis $\mathcal{B} = \{1 + jt : j \in J\}$. That makes $\mathrm{NK}_0(R,\alpha)$ a finitely generated $\mathbb{Z}[\mathbb{N}]$-module.

3 Waldhausen's Nil-Groups

In the calculation of the K-theory of Type (2) virtually infinite cyclic groups, the Nil-groups of Waldhausen [23], [24] appear. We recall their definition. Let $\mathcal{T}$ be the category of triples $\mathbf{R} = (R; B_0, B_1)$, where B_i, $i = 0, 1$ are two R-bimodules. A morphism in $\mathcal{T}$ is a triple

$$(\phi, f_0, f_1)\colon (R; B_0, B_1) \to (S, C_0, C_1)$$

where $\phi\colon R \to S$ is a ring homomorphism and $f_i\colon B_i \otimes_R S \to C_i$, $i = 0, 1$, are $R - S$-bimodule homomorphisms. Waldhausen's Nil-groups are functors from the category $\mathcal{T}$ to abelian groups. For an object $\mathbf{R}$ in $\mathcal{T}$, we first define an exact category $\mathcal{N}il(\mathbf{R})$ with objects quadruples $(P, Q; p, q)$, where P and Q are finitely generated projective right R-modules and

$$p\colon P \to Q \otimes B_0, \quad q\colon Q \to P \otimes B_1$$

is a pair of R-homomorphisms such that the compositions

$$(*) \qquad \begin{array}{l} P \xrightarrow{p} Q \otimes B_0 \xrightarrow{q\otimes 1} P \otimes B_1 \otimes B_0 \cdots \\ Q \xrightarrow{q} P \otimes B_1 \xrightarrow{p\otimes 1} P \otimes B_0 \otimes B_1 \cdots \end{array}$$

are zero after finitely many steps. Morphisms are homomorphisms on the modules that are compatible with the maps. Exact sequences are defined the obvious way.

Notice that there is a forgetful functor $\phi\colon \mathcal{N}il(\mathbf{R}) \to \mathcal{P}_R \times \mathcal{P}_R$, where $\mathcal{P}_R$ is the category of finitely generated projective right R-modules. Then

$$\widetilde{\mathrm{Nil}}_i^W(\mathbf{R}) = \ker\Big(K_i\big(\mathcal{N}il(\mathbf{R})\big) \xrightarrow{\phi_i} K_i(\mathcal{P}_R \times \mathcal{P}_R)\Big), \quad \text{for } i \in \mathbb{Z}.$$

For the calculation of Waldhausen's Nil-groups we will use the interpretation given in [6], [16] and [17]. The basic idea is that the calculation of lower Waldhausen's Nil-groups is reduced to the reduced K-theory of an augmented ring. For an R-bimodule M, $T_R(M)$ denotes the tensor algebra of M, *i.e.*, the graded R-algebra which admits a splitting

$$T_R(M) = R \oplus M \oplus (M \otimes_R M) \oplus (M \otimes_R M \otimes_R M) \oplus \cdots$$

as an R-bimodule and with multiplication by concatenation on terms of positive degree. The construction involves a functor

$$\rho\colon \mathcal{T} \to \mathrm{Rings}, \quad \rho(\mathbf{R}) = R_\rho = \begin{pmatrix} T_R(B_1 \otimes_R B_0) & B_1 \otimes_R T_R(B_0 \otimes_R B_1) \\ B_0 \otimes_R T_R(B_1 \otimes_R B_0) & T_R(B_0 \otimes_R B_1) \end{pmatrix}$$

with multiplication given as matrix multiplication and on each entry by concatenation. There is an alternate description of the ring R_ρ: In [6], there is a construction of a twisted polynomial extension category associated to $\mathbf{R}$. The resulting category admits a basic object $b = (R, R)$, *i.e.*, for each v in the category there is an object u such that $u \oplus v$ is isomorphic to b^k [2, p. 197]. The ring R_ρ is the endomorphism ring of the basic object in the twisted polynomial extension [17, Proposition 2.3]. The basic result is that the lower Waldhausen's Nil-groups associated to the triple $\mathbf{R}$ can be computed using the K-theory of the ring R_ρ. More specifically, there is a natural augmentation map

$$R_\rho \xrightarrow{\varepsilon} \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}.$$

We write $\mathrm{NK}_i(\mathbf{R})$ for the kernel of the map induced by ε on K_i-groups. Then there is a natural isomorphism between the NK-groups and Waldhausen's Nil-groups ([6, Proposition 2.6] for $i = 1$, and [17, the remarks following Theorem 3.7] for $i < 1$)

$$\mathrm{NK}_i(\mathbf{R}) \cong \widetilde{\mathrm{Nil}}_{i-1}^W(\mathbf{R}), \quad \text{for } i \le 1.$$

Thus vanishing results of Waldhausen's Nil-groups can be applied to the NK-groups.

We now formulate the Nil-groups of interest. Let Γ be a virtually infinite cyclic group of Type (2). Then Γ admits a decomposition as an amalgamated free product of finite groups

$$\Gamma = G_0 *_H G_1, \quad [G_0 : H] = [G_1 : H] = 2.$$

Then the Waldhausen's Nil-groups are defined using the triple $(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_1 - H])$. Choosing right coset representatives, we see that the $\mathbb{Z}H$-bimodules $\mathbb{Z}[G_i - H]$, $i = 0, 1$, are isomorphic to $\mathbb{Z}H$ as left $\mathbb{Z}$-modules, but the right action is twisted by an automorphism $\mathbb{Z}H$, induced by an automorphism of H.

We can now formulate the general setup. Let R be a ring with one and $\alpha\colon R \to R$ a ring automorphism. We denote by R^α the $R-R$-bimodule which is R as a left R-module but the right multiplication is given by: $x.r = x\alpha(r)$. Our objective is to study the lower Waldhausen's Nil-groups of the form $\widetilde{\mathrm{Nil}}_*^W(R; R^\alpha, R^\beta)$ where α and β are automorphisms of R. From now on, $\otimes$ means $\otimes_R$.

We start with a basic algebraic observation.

Lemma 3.1 *With the above notation, there is an R-bimodule isomorphism*

$$i_{\alpha,\beta}\colon R^\alpha \otimes R^\beta \to R^{\alpha\beta}.$$

Proof The isomorphism is given by

$$i_{\alpha,\beta}(x \otimes y) = x\alpha(y). \qquad \blacksquare$$

Proposition 3.2 *There is an isomorphism*

$$\iota\colon \widetilde{\mathrm{Nil}}_i^W(R; R^\alpha, R^\beta) \to \widetilde{\mathrm{Nil}}_i^W(R; R^{\alpha\beta}, R), \quad \textit{for all } i \in \mathbb{Z}.$$

Proof We write $\mathbf{R} = (R; R^\alpha, R^\beta)$ and $\mathbf{R}' = (R; R^{\alpha\beta}, R)$. We will define an equivalence of the Nil-categories

$$\iota\colon \mathcal{N}il(\mathbf{R}) \to \mathcal{N}il(\mathbf{R}').$$

On objects, we define

$$\iota(P, Q; p, q) = \left(P \otimes R^\beta, Q; (1_Q \otimes i_{\alpha,\beta}) \circ (p \otimes 1_{R^\beta}), q\right)$$

(for every right R-module M we will always identify $M \otimes R$ with M in the natural way). On morphisms ι is defined by

$$\iota(f, g) = (f \otimes 1_{R^\beta}, g).$$

The functor ι is exact because R^β is a free (and thus flat) left R-module.

We can now describe the inverse of ι. That will be an exact functor

$$\iota'\colon \mathcal{N}il(\mathbf{R}') \to \mathcal{N}il(\mathbf{R}).$$

For a quadruple $(P', Q'; p', q')$ with

$$p'\colon P' \to Q' \otimes R^{\alpha\beta}, \quad q'\colon Q' \to P'$$

we define

$$\iota'(P', Q'; p', q') = \left(P' \otimes R^{\beta^{-1}}, Q'; (1_Q \otimes i_{\alpha\beta,\beta^{-1}}) \circ (p' \otimes 1_{R^{\beta^{-1}}}), (1_{P'} \otimes i_{\beta^{-1},\beta} \circ q')\right).$$

For the same reasons as ι, ι' defines an exact functor on the $\mathcal{N}il$-categories.

It is immediate to check that ι and ι' define an equivalence between the two categories. Thus ι induces an isomorphism on the K-groups of the $\mathcal{N}il$-categories. The result follows. $\blacksquare$

Remark 3.3 Actually the proof of Proposition 3.2 shows that there are isomorphisms for all $i \in \mathbb{Z}$:

$$\widetilde{\mathrm{Nil}}_i^W(R; R^\alpha, R^\beta) \cong \widetilde{\mathrm{Nil}}_i^W(R; R^{\alpha\beta}, R) \cong \widetilde{\mathrm{Nil}}_i^W(R; R, R^{\alpha\beta}).$$

Corollary 3.4 *There is an isomorphism*

$$\mathrm{NK}_i(R; R^\alpha, R^\beta) \cong \mathrm{NK}_i(R; R^{\alpha\beta}, R), \quad \textit{for } i \leq 1.$$

Proof The lower Waldhausen's Nil-groups and the lower NK-groups are isomorphic. ∎

Because of Corollary 3.4, we will concentrate on triples of the form $\mathbf{R} = (R; R^\alpha, R)$ where α is an automorphism of R. In this case, we will first try to give a more manageable description of the ring R_ρ. If we set $B_0 = R^\alpha$, $B_1 = R$, then the ring R_ρ takes the form

$$\begin{aligned} R_\rho &= \begin{pmatrix} T_R(R \otimes_R R^\alpha) & R \otimes_R T_R(R^\alpha \otimes_R R) \\ R^\alpha \otimes_R T_R(R \otimes_R R^\alpha) & T_R(R^\alpha \otimes_R R) \end{pmatrix} \\ &= \begin{pmatrix} T_R(R \otimes_R R^\alpha) & T_R(R \otimes_R R^\alpha) \otimes_R R \\ T_R(R^\alpha \otimes_R R) \otimes_R R^\alpha & T_R(R^\alpha \otimes_R R) \end{pmatrix}. \end{aligned}$$

We will identify R_ρ with a subring of the ring of 2×2-matrices over a twisted polynomial ring. The automorphism α induces an automorphism on the polynomial ring $R[s]$, denoted α, with $\alpha(s) = s$. Let $R_\alpha^{(2)}[s,t] = (R[s])_\alpha[t]$. Then $R_\alpha^{(2)}[s,t]$ is the polynomial ring on two commuting variables which is isomorphic to $R[s,t]$ as a left R-module. The right action of R on the variables s and t is given by:

$$s.r = rs, \quad t.r = \alpha(r)t, \quad \text{for all } r \in R$$

(the superscript $^{(2)}$ in the notation suggests that the twist, by α, occurs in the second variable). We will describe R_ρ as a subring of $M_2(R_\alpha^{(2)}[s,t])$. Set

$$R_{\alpha,\mathrm{d}}^{(2)}[s,t] = \{p \in R_\alpha^{(2)}[s,t] : \mathrm{degree}_s(m) = \mathrm{degree}_t(m),\ m \text{ a monomial of } p\}$$

for the subring of $R_\alpha^{(2)}[s,t]$ generated by all the monomials which have equal degrees with respect to the variables.

Proposition 3.5 *There is a ring isomorphism*

$$\mu\colon R_\rho \to R'_\rho = \begin{pmatrix} R_{\alpha,d}^{(2)}[s,t] & R_{\alpha,d}^{(2)}[s,t]s \\ R_{\alpha,d}^{(2)}[s,t]t & R_{\alpha,d}^{(2)}[s,t] \end{pmatrix}.$$

Proof We describe how to define μ on each entry of R_ρ. It is enough to show how to define μ on each component of an entry of R_ρ. We write 1 for the identity element of R and 1_α for the generator of R_α corresponding to 1. We map 1 to s, 1_α to t and tensor products of 1 and 1_α to corresponding products of s and t. Now every element in a component of an entry of R_ρ can be represented in the form rp, where $r \in R$ and p is a tensor product of 1's and 1_α's. We define $\mu(rp) = r\mu(p)$. The inverse map is defined similarly. We leave it to the reader to show that μ is a ring isomorphism. ■

Remark 3.6 There is a natural comment to be made for the result of Proposition 3.5. For the characterization of the ring R_ρ we used the original definition of Waldhausen's Nil-groups. It is natural to ask if the same result can be obtained using a different ring from R_ρ to begin with. The authors are not aware of such a construction at this point.

There is a natural augmentation map

$$\varepsilon' \colon R'_\rho \to \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$$

such that $\mu \circ \varepsilon = \varepsilon'$ where ε is the corresponding augmentation map on R_ρ. The following result is a consequence of Proposition 3.5.

Corollary 3.7 *The groups* $\mathrm{NK}_i(\mathbf{R})$, $i \leq 1$, *are isomorphic to the kernel of the map induced by the augmentation* ε'*:*

$$\mathrm{NK}_i(\mathbf{R}) = \ker\left(K_i(R'_\rho) \xrightarrow{\varepsilon'_i} K_i(R \times R)\right), \quad i \leq 1.$$

Proposition 3.8 *If R is regular then* $\mathrm{NK}_i(R; R^\alpha, R) = 0$ *for all* $i \leq 1$. *In particular, the augmentation maps* ε_i *and* ε'_i *induce an isomorphisms on lower* K*-groups.*

Proof For $i \leq 1$, the groups $\mathrm{NK}_i(R; R^\alpha, R)$ are isomorphic to Waldhausen's groups $\widetilde{\mathrm{Nil}}^W_{i-1}(R; R^\alpha, R)$ which vanish for R regular [23], [24]. The result follows. ■

Let J be a two-sided ideal of R which is invariant under the automorphism α. Then the projection $R \to R/J$ induces a morphism of triples

$$(R; R^\alpha, R) \to \left(R/J; (R/J)^\alpha, R/J\right)$$

and thus a ring homomorphism

$$\chi_J \colon R'_\rho \to (R/J)'_\rho.$$

Lemma 3.9 *Using the above notation,*

$$(R/J)'_\rho \cong R'_\rho / J'_\rho$$

where J'_ρ is the two-sided ideal of R'_ρ generated by $J \times J$, i.e.,

$$J'_\rho = \left\langle \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \right\rangle.$$

Proof The proof is the same as in [17, Proposition 2.10]. ■

We also denote by $\chi_J\colon R'_\rho \to R'_\rho/J'_\rho$ the natural projection homomorphism.

Remark 3.10 With the setting as in Lemma 3.9,

$$J'_\rho = \begin{pmatrix} J^{(2)}_{\alpha,\mathrm{d}}[s,t] & J^{(2)}_{\alpha,\mathrm{d}}[s,t]s \\ J^{(2)}_{\alpha,\mathrm{d}}[s,t]t & J^{(2)}_{\alpha,\mathrm{d}}[s,t] \end{pmatrix}.$$

In particular, if J is nilpotent J'_ρ is also nilpotent.

We are interested in determining the properties of the map that is induced in K-theory by χ_J.

Proposition 3.11 *Let J be a nilpotent two-sided ideal of R which is invariant under α. Then the ring homomorphism χ_J, described above, induces an isomorphism in K_i-groups, for $i \leq 0$.*

Proof Using Lemma 3.9, the map χ_J can be described as a quotient map

$$\chi_J\colon R'_\rho \to R'_\rho/J'_\rho.$$

But the ideal J'_ρ is a nilpotent ideal of R'_ρ (Remark 3.10). The result follows from [1, Proposition 2.12] (also [17, Proposition 2.11]). ■

Corollary 3.12 *Let R be an Artinian ring and J its Jacobson radical. Then the quotient map χ_J induces isomorphism on K_i-groups for $i \leq 0$.*

Proof Since R is Artinian, J is nilpotent. Also, J is invariant of every automorphism of R. The result follows from Proposition 3.11. ■

Corollary 3.13 *Let R be an α-quasi-regular ring. Then $\mathrm{NK}_i(R; R^\alpha, R) = 0$, for all $i \leq 0$.*

Proof Let J be a two-sided, α-invariant nilpotent ideal of R such that the ring R/J is regular. By Proposition 3.11, the map

$$(\chi_J)_i\colon K_i(R'_\rho) \to K_i\big((R/J)'_\rho\big)$$

is an isomorphism for $i \leq 0$. There is a commutative diagram

$$\begin{tikzcd} K_i(R'_\rho) \arrow[r, "(\chi_J)_i"] \arrow[d, "\varepsilon'_i"'] & K_i\big((R/J)'_\rho\big) \arrow[d, "(\varepsilon'_J)_i"] \\ K_i(R \times R) \arrow[r] & K_i(R/J \times R/J) \end{tikzcd}$$

where the vertical maps are induced by the augmentations. Since R/J is regular, ε'_J is an isomorphism (Proposition 3.8). Also, since J is nilpotent the bottom map is an isomorphism. Thus ε' is an isomorphism and $\mathrm{NK}_i(R; R^\alpha, R) = 0$ for $i \leq 0$. ■

Now we specialize to the case that $R = \mathbb{Z}H$, the group ring of a finite group H of order n. Let α be an automorphism of R induced by an automorphism of H. Choose a hereditary order $\mathcal{M}$ as in the introduction. Then we can define triples in $\mathcal{T}$,

$$\mathbf{M} = (\mathcal{M}; \mathcal{M}^{\alpha}, \mathcal{M}),$$

$$\mathbf{R}/n\mathbf{M} = \left(R/n\mathcal{M}; (R/n\mathcal{M})^{\alpha}, R/n\mathcal{M}\right),$$

$$\mathbf{M}/n\mathbf{M} = \left(\mathcal{M}/n\mathcal{M}; (\mathcal{M}/n\mathcal{M})^{\alpha}, \mathcal{M}/n\mathcal{M}\right)$$

(we still use α to denote the homomorphism induced by α on the other rings involved in the calculation). The triples determine matrix rings (Proposition 3.5):

$$\mathcal{M}'_{\rho} \cong \mathcal{M}_{\rho} \text{ corresponds to } (\mathcal{M}; \mathcal{M}^{\alpha}, \mathcal{M})$$

$$(R/n\mathcal{M})'_{\rho} \cong (R/n\mathcal{M})_{\rho} \text{ corresponds to } \left(R/n\mathcal{M}; (R/n\mathcal{M})^{\alpha}, R/n\mathcal{M}\right)$$

$$(\mathcal{M}/n\mathcal{M})'_{\rho} \cong (\mathcal{M}/n\mathcal{M})_{\rho} \text{ corresponds to } \left(\mathcal{M}/n\mathcal{M}; (\mathcal{M}/n\mathcal{M})^{\alpha}, \mathcal{M}/n\mathcal{M}\right).$$

There is a commutative diagram of matrix rings

$$(**) \qquad \begin{array}{ccc} R'_{\rho} & \xrightarrow{i} & \mathcal{M}'_{\rho} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ (R/n\mathcal{M})'_{\rho} & \xrightarrow{j} & (\mathcal{M}/n\mathcal{M})'_{\rho} \end{array}$$

where i and j are induced by the inclusions and p and q by the projections. Actually, the maps in the diagram are induced by the maps of triples but this is not of use to us.

Proposition 3.14 *The diagram $(**)$ is a pull-back square of rings.*

Proof By construction, the sequence

$$0 \to R'_{\rho} \xrightarrow{(i,-q)} \mathcal{M}'_{\rho} \oplus (R/n\mathcal{M})'_{\rho} \xrightarrow{(p,j)} (\mathcal{M}/n\mathcal{M})'_{\rho} \to 0$$

is an exact sequence of R'_{ρ}-bimodules because it is exact on each entry of the matrix representation of the rings. The result follows. ■

Remark 3.15 There are also the classical pull-back diagrams

$$\begin{array}{ccc} R & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \\ R/n\mathcal{M} & \longrightarrow & \mathcal{M}/n\mathcal{M} \end{array} \qquad \begin{array}{ccc} R \times R & \longrightarrow & \mathcal{M} \times \mathcal{M} \\ \downarrow & & \downarrow \\ R/n\mathcal{M} \times R/n\mathcal{M} & \longrightarrow & \mathcal{M}/n\mathcal{M}. \end{array}$$

The result of Proposition 3.14 allows us to use Mayer-Vietoris type exact sequences for the calculation of the Nil-groups.

Proposition 3.16 *There is an epimorphism*

$$\partial\colon\ \mathrm{NK}_1\big(\mathcal{M}/n\mathcal{M};(\mathcal{M}/n\mathcal{M})^{\alpha},\mathcal{M}/n\mathcal{M}\big)\ \to \mathrm{NK}_0(R;R^{\alpha},R).$$

Proof Combining the Mayer-Vietoris sequences for the diagram $(**)$ and the second diagram in Remark 3.15, we obtain a commutative diagram of exact sequences

$$\begin{array}{ccccc}
K_1\big((\mathcal{M}/n\mathcal{M})'_\rho\big) & \longrightarrow & K_0(R'_\rho) & \longrightarrow & K_0\big((R/n\mathcal{M})'_\rho\big)\oplus K_0(\mathcal{M}'_\rho) \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
K_1(\mathcal{M}/n\mathcal{M}\times\mathcal{M}/n\mathcal{M}) & \longrightarrow & K_0(R\times R) & \longrightarrow & K_0(R/n\mathcal{M}\times R/n\mathcal{M}) \\
 & & & & \oplus K_0(\mathcal{M}\times\mathcal{M})
\end{array}$$

where the vertical maps are introduced by the augmentations, and thus they are split epimorphisms. Using Corollary 3.7, we see that taking the kernels of the vertical maps induces an exact sequence

$$\mathrm{NK}_1(\mathbf{M}/n\mathbf{M}) \xrightarrow{\partial} \mathrm{NK}_0(\mathbf{R}) \to \mathrm{NK}_0(\mathbf{R}/n\mathbf{M})\oplus\mathrm{NK}_0(\mathbf{M}).$$

But the ring R/nM is Artinian and thus α-quasi-regular (Remark 2.3, Corollary 3.13) and $\mathcal{M}$ is regular. Thus the two NK-groups at the right end of the sequence vanish. The result follows. ∎

The rest of the calculation of the exponent of the Nil-groups follows closely the methods used in [5] and [17]. We will outline the method of the proof. Remember that $R=\mathbb{Z}H$ and the order of the finite group H is n, and n' is as in the introduction.

Theorem 3.17 *The group* $\mathrm{NK}_0(R;R^{\alpha},R)$ *has exponent* n'*, i.e.,*

$$n'\,\mathrm{NK}_0(R;R^{\alpha},R)=0.$$

Proof By Proposition 3.16, we need to calculate the exponent of the generators of the group $\mathrm{NK}_1\big(\mathcal{M}/n\mathcal{M};(\mathcal{M}/n\mathcal{M})^{\alpha},\mathcal{M}/n\mathcal{M}\big)$. The method of proof of Proposition 4.9 in [17] applies here because J'_ρ is a nilpotent ideal (Remark 3.10). Let I_ρ be the augmentation ideal

$$I_\rho=\ker\big(\varepsilon'\colon(\mathcal{M}/n\mathcal{M})'_\rho\to\mathcal{M}/n\mathcal{M}\times\mathcal{M}/n\mathcal{M}\big).$$

Let $\widetilde{J}_\rho = J'_\rho\cap I_\rho$. Then in Proposition 4.9 in [17], it was shown that there is an epimorphism

$$1+\widetilde{J}_\rho\to\mathrm{NK}_1(\mathbf{M}/n\mathbf{M}).$$

Thus a set of generators of $\mathrm{NK}_1(\mathbf{M}/n\mathbf{M})$ can be defined by the image of a set of generators of $1+\widetilde{J}_\rho$. Again, interpreting the result in Proposition 4.9 in [17], a set of generators of $1+\widetilde{J}_\rho$ is given by element of the form

$$\begin{pmatrix}1+p_{1,1} & 0\\ 0 & 1+p_{2,2}\end{pmatrix},\quad \begin{pmatrix}1 & p_{1,2}\\ 0 & 1\end{pmatrix},\quad \begin{pmatrix}1 & 0\\ p_{2,1} & 1\end{pmatrix}$$

where $p_{1,1}$ and $p_{2,2}$ are in $J^{(2)}_{\alpha,\mathrm{d}}[s,t]$, $p_{1,2}\in J^{(2)}_{\alpha,\mathrm{d}}[s,t]s$, and $p_{2,1}\in J^{(2)}_{\alpha,\mathrm{d}}[s,t]t$. The exponents of these elements is n' [17, Theorem 5.1]. ∎

Then it is immediate that the same result applies to Waldhausen's Nil-groups.

Theorem 3.18 *The Waldhausen's* NK_0*-groups that appear in the decomposition of a virtually infinite cyclic group with base H of order n have exponent* n'.

4 Examples

We present a class of examples where we calculate the exponent of the reduced K-group of a group of Type (2). The lower Nil-groups of Waldhausen vanish when the base group is a cyclic group of prime order [16]. Recently, it has been announced that the vanishing result holds when the order of the group is not divisible by a square [8].

In our examples, the base group will be the cyclic group, $C(2^k)$ of order 2^k. For an integer n, let ζ_n denote the primitive n-th root of unity and $\mathbb{Z}[\zeta_n]$ the ring of cyclotomic integers. If p is a prime, we choose the roots of unity so that $\zeta_{p^{k+1}}^p = \zeta_{p^k}$. Set $n = 2^{k'}$ where $k' = \log_2(k+1) + k + 1$, the number corresponding to n', in the Main Theorem, for $C(2^{k+1})$. Then

$$\mathbb{Z}\left[\frac{1}{2}\right] = \mathbb{Z}\left[\frac{1}{n}\right].$$

In the following lemmas we summmarize some of the results for the $\tilde{\mathrm{K}}_0(\mathbb{Z}G)$ for G certain groups of order 2^{k+1}.

Lemma 4.1 *For the cyclic groups of order* 2^{k+1},

$$\tilde{K}_0\big(\mathbb{Z}C(2^{k+1})\big) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \cong \bigoplus_{i=1}^{k+1} \tilde{K}_0(\mathbb{Z}[\zeta_{2^i}]).$$

Furthermore, if $\iota\colon C(2^{k+1}) \to C(2^{k+2})$ *is the inclusion map, then the map*

$$\iota_* \otimes \mathrm{id}\colon \tilde{K}_0\big(\mathbb{Z}C(2^{k+1})\big) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to \tilde{K}_0\big(\mathbb{Z}C(2^{k+2})\big) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$$

is injective and has the form

$$\iota'_* = \big((\iota_1)_*, \ldots, (\iota_{k+1})_*\big)\colon \bigoplus_{i=1}^{k+1} \tilde{K}_0(\mathbb{Z}[\zeta_{2^i}]) \to \bigoplus_{i=1}^{k+2} \tilde{K}_0(\mathbb{Z}[\zeta_{2^i}])$$

where $\iota_j\colon \mathbb{Z}[\zeta_{2^j}] \to \mathbb{Z}[\zeta_{2^{j+1}}]$ *is the natural map* ($j = 1, \ldots, k+1$).

Proof For the first isomorphism, notice that Theorem 1.2 in [14] shows that ι_* is an epimorphism with kernel a 2-group. The $\tilde{\mathrm{K}}_0$-groups of the rings of integers are isomorphic to their class groups which have odd order [21]. The result follows. Propositions 2.1 and 2.2 in [22] imply that the following diagram commutes

$$\begin{tikzcd}
\tilde{\mathrm{K}}_0\big(\mathbb{Z}C(2^{k+1})\big) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \arrow[r, "\iota_* \otimes \mathrm{id}"] \arrow[d, "\cong"'] & \tilde{\mathrm{K}}_0\big(\mathbb{Z}C(2^{k+2})\big) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \arrow[d, "\cong"] \\
\bigoplus_{i=1}^{k+1} \tilde{\mathrm{K}}_0(\mathbb{Z}[\zeta_{2^i}]) \arrow[r, "\iota'_*"] & \bigoplus_{i=1}^{k+2} \tilde{\mathrm{K}}_0(\mathbb{Z}[\zeta_{2^i}]).
\end{tikzcd}$$

That completes the proof of the second part. ∎

Lemma 4.2 *Let G be $D(2^{k+2})$, the dihedral group of order 2^{k+2} or $Q(2^{k+2})$, the quartenionic group of order 2^{k+2}. Let $\iota\colon C(2^{k+1}) \to G$ be the inclusion map. Then*

$$\iota_* \otimes \mathrm{id}\colon \tilde{K}_0\big(\mathbb{Z}C(2^{k+1})\big) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \to \tilde{K}_0(\mathbb{Z}G) \otimes \mathbb{Z}\left[\frac{1}{2}\right]$$

is an epimorphism.

Proof By the main result in [13], there is an isomorphism

$$\tilde{\mathrm{K}}_0(\mathbb{Z}G) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \cong \bigoplus_{i=1}^{k+1} \tilde{\mathrm{K}}_0(\mathbb{Z}[\zeta_{2^i} + \bar{\zeta}_{2^i}]).$$

The class numbers of the rings of integers appearing in the decomposition divide the class numbers of the corresponding cyclotomic rings. Thus they are odd [21]. The methods of Sections 3 and 4 of [4] apply to this case also and show that the map $\iota_* \otimes \mathrm{id}$ is given by the norm map and it is an epimorphism. ∎

Theorem 4.3 *Let $\Gamma = G_0 *_{C(2^{k+1})} G_1$ with $G_j = C(2^{k+2}), Q(2^{k+2}), D(2^{k+2})$, for $j = 0, 1$. Then $\tilde{K}_0(\mathbb{Z}G)$ can be computed from the class groups of cyclotomic integers. More precisely,*

(i) *If $G_j = C(2^{k+2})$, $j = 0, 1$*

$$\tilde{K}_0(\mathbb{Z}\Gamma) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \cong \bigoplus_{i=1}^{k+1} \mathrm{coker}\big((\iota_i)_*, (\iota_i)_*\big)$$

(ii) *If $G_0 = C(2^{k+2})$, $G_1 = D(2^{k+2})$ or $Q(2^{k+2})$, then*

$$\tilde{K}_0(\mathbb{Z}\Gamma) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \cong \bigoplus_{i=1}^{k+1} \mathrm{coker}\big((\iota_i)_*, N_i\big)$$

where $N_i\colon \tilde{K}_0(\mathbb{Z}[\zeta_{2^i}]) \to \tilde{K}_0(\mathbb{Z}[\zeta_{2^i} + \bar{\zeta}_{2^i}])$ denotes the norm map.

(iii) *If G_j is a dihedral or quartenionic 2-group,* $j = 0, 1$,

$$\tilde{K}_0(\mathbb{Z}\Gamma) \otimes \mathbb{Z}\left[\frac{1}{2}\right] \cong \bigoplus_{i=1}^{k+1} \tilde{K}_0(\mathbb{Z}[\zeta_{2^i} + \bar{\zeta}_{2^i}]).$$

Proof Taking reduced groups in the Waldhausen's exact sequence, we have an exact sequence

$$\tilde{\mathrm{K}}_0\big(\mathbb{Z}C(2^{k+1})\big) \xrightarrow{(s_0,s_1)} \tilde{\mathrm{K}}_0(\mathbb{Z}G_0) \oplus \tilde{\mathrm{K}}_0(\mathbb{Z}G_1) \to \tilde{\mathrm{K}}_0(\mathbb{Z}\Gamma)/\mathcal{N} \to K_{-1}\big(\mathbb{Z}C(2^{k+1})\big)$$

where the s_j are induced by the inclusion map, and $\mathcal{N} = \mathrm{NK}_{-1}\big(\mathbb{Z}C(2^{k+1}); \mathbb{Z}[G_0 - C(2^{k+1})], \mathbb{Z}[G_1 - C(2^{k+1})]\big)$. The group $K_{-1}\big(\mathbb{Z}C(2^{k+1})\big) = 0$ [3] and $\mathcal{N} \otimes \mathbb{Z}[1/2] = 0$ (Theorem 3.18). Then the result follows from Lemmas 4.1 and 4.2. ∎

Corollary 4.4 *Let* $\Gamma = D_8 *_{C(4)} D_8$. *Then*

$$\tilde{K}_0(\mathbb{Z}\Gamma) \otimes \mathbb{Z}\left[\frac{1}{2}\right] = 0.$$

Remark 4.5 Actually in the last example

$$\tilde{\mathrm{K}}_0(\mathbb{Z}\Gamma) \cong \widetilde{\mathrm{Nil}}^W_{-1}(\mathbb{Z}[\mathbb{Z}/4\mathbb{Z}]; \mathbb{Z}[D_8 - \mathbb{Z}/4\mathbb{Z}], \mathbb{Z}[D_8 - \mathbb{Z}/4\mathbb{Z}]),$$

which has exponent 8.

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