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ORIGINAL PAPER

A Bass–Heller–Swan formula for pseudoisotopies

L. Christine Kinsey · Stratos Prassidis

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Abstract The Bass–Heller–Swan formula is a basic calculational tool in pseudoisotopy K-theory. We describe the Nil-groups and the Bass–Heller–Swan splitting for the group of the pseudoisotopies of a closed manifold. We use the methods of controlled topology used in the Bass–Heller–Swan splitting in K-theory.

Keywords Pseudoisotopy · Relaxation · Nil-groups

Mathematics Subject Classification (2010) 57N37 · 19D10

1 Introduction

The Bass-Heller–Swan Formula [2,3] is one of the most important calculational tools in algebraic K-theory. For any ring R,

$$K_1(R[t,t^{-1}]) \cong K_1(R)\oplus K_0(R)\oplus\widetilde{Nil}(R)\oplus\widetilde{Nil}(R).$$

In more geometric terms, the formula can be written, for a space X, as

$$Wh(X\times S^1) \cong Wh(X)\oplus\tilde{K}_0(X)\oplus\widetilde{Nil}(X)\oplus\widetilde{Nil}(X).$$

In [23], the second formula was generalized to Whitehead spaces and it was proved using methods from controlled topology [9,11].

Dedicated to Bruce Williams for his sixtieth birthday

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L. C. Kinsey · S. Prassidis (✉)
Department of Mathematics, Canisius College, Buffalo, NY 14208, USA
e-mail: prasside@canisius.edu

L. C. Kinsey
e-mail: kinsey@canisius.edu

In this paper we consider only smooth pseudoisotopies. There is a connection between the Whitehead space and stable pseudoisotopy space given by [1,6–8,18,20]:

$$\Omega\mathcal{W}h(X) \simeq \mathbb{P}(X).$$

Thus, there is a Bass–Heller–Swan splitting for the space of stable pseudoisotopies. Such a formula for Hilbert cube manifolds was given in [23]. Unfortunately, the construction of the homotopy equivalence between the Whitehead space and the pseudoisotopy space involves many choices, at least in the topological category. This makes the transfer of explicit constructions from the Whitehead spaces to the pseudoisotopy spaces cumbersome. That is why a direct calculation for the Bass–Heller–Swan formula for pseudoisotopies is desirable. Also a splitting in A-theory is given in [28]. Our methods adapt the methods in [23] and they are more geometric than the ones in [28].

The pseudoisotopy group of a space is the group of path components of the pseudoisotopy space. In this paper, we give explicit constructions of the pseudoisotopy Nil-groups and a geometric splitting of the pseudoisotopy group over S^1.

Theorem (*Bass–Heller–Swan Splitting for Pseudoisotopy Groups*) *Let X be a closed smooth manifold.*

(1) *There is a splitting of abelian groups*

$$\mathcal{P}(X\times S^1) \cong \mathcal{RP}(X\times S^1)\oplus\mathcal{NP}(X) \cong \mathcal{P}(X)\oplus\mathcal{P}_b(X\times\mathbb{R})\oplus\mathcal{NP}(X),$$

where $\mathcal{RP}(X\times S^1)$ is the subgroup of the relaxed pseudoisotopies.

(2) *Furthermore, if $dim(X) \geq 7$ and $\Sigma : \mathcal{P}(X\times S^1) \to \mathcal{P}(X\times S^1\times I)$ is the suspension map, then there is a group isomorphism*

$$\Sigma\mathcal{NP}(X) \cong \mathcal{N}_+\mathcal{P}(X)\oplus\mathcal{NP}_-(X)$$

and the two summands are isomorphic.

(3) *If $n \geq 7$ and $\mathbb{P}(-)$ denotes the stable pseudoisotopy group then there is a group isomorphism*

$$\mathbb{P}(X\times S^1) \cong \mathbb{P}(X)\oplus\mathbb{P}_b(X\times\mathbb{R})\oplus\mathcal{N}\mathbb{P}_+(X)\oplus\mathcal{N}\mathbb{P}_-(X).$$

Taking bounded pseudoisotopies over the reals corresponds to delooping. Thus, $\mathcal{P}_b(X\times\mathbb{R}) \cong Wh(X\times I)$ [1]. The dimension assumption comes from the stable range estimates for pseudoisotopies ([14,15,30,29] for the smooth case and [5] for the topological case, also [36] for a review and applications).

In some sense, controlled topology is used in the proof of the Main Theorem. More specifically, the methods used combine the explicit construction of the pseudoisotopy relaxation given in [31] and the controlled methods used in [10,19,21,25,26]. Using the construction in [31], we give an explicit calculation of the relaxation map:

$$r : \mathcal{P}(X\times S^1) \to \mathcal{RP}(X\times S^1)$$

that splits the "forget-control" map. The splitting is constructed through a different summand of the pseudoisotopy group, the relaxed pseudoisotopies. Actually, the constructions in [31] provide us also with a model for the Nil-components of the pseudoisotopy group. The problem of splitting the Nil-part into two components was stated explicitly in [6]. Results in this direction are given in [4]. Also, it should be noted that splitting results for higher K-groups were given in [16].

In the topological category the theorem holds as well when the dimension of the manifold is greater than or equal to the stable range. Furthermore, in this case the relaxed subgroup is isomorphic to the controlled subgroup $\mathcal{P}_c^{\text{top}}(X \times S^1 \to S^1)$ (Section 8).

Our main Theorem should be thought as a version of the Bass–Heller–Swan splitting in Wh_2 [4,16]. Different versions of this formula were given in [4,6] for pseudoisotopies, in [13,23] in higher K-theory and in [28] in A-theory. Geometrically, the Bass–Heller–Swan formula may be thought as characterizing splitting obstructions "over S^1". As such it was given in [12,23,24,31,35]. The splitting theorem can be considered as a geometric version of the higher fibering problem over the circle that was introduced by Farrell in [12] and was generalized in [31].

The main result of this paper gives partial results to the question BW1 in [22], providing a geometric description of the π_0 of the Nil-term. Furthermore that question motivated the material in Sects. 7 and 8. The second part of our main theorem was motivated by question BW2 in [22]. We give a splitting of the Nil-term into two isomorphic pieces. In Sects. 7 and 8, we give an alternate description of the $\mathcal{NP}$-group. The authors believe that embeddings can be used in an alternate definition of the Nil-group [6].

Part (1) in the Main Theorem holds true if we consider bounded pseudoisotopies over $\mathbb{R}^m$. Parts (2) and (3) depend on some algebraic results on the pseudoisotopy group and they do not generalize immediately to the bounded setting. For parts (2) and (3), we need the dimensional assumption to ensure that the suspension map is an isomorphism.

This is the first in a series of papers that will provide splittings not only of the pseudoisotopy group but also of the pseudoisotopy space. That generalization will give complete answers to questions BW1 and BW2 in [22]. Also, the authors intend to get similar splittings for compact Hilbert cube manifolds using the methods in [9,11,17,20,23]. Even though in this paper we restrict ourselves to finite dimensional manifolds, many of the ideas come from and extend to Hilbert cube manifolds. As a continuation of this work, the authors will consider the splitting theorem for twisted products, i.e., for manifolds X that come equipped with an approximate fibration to S^1 which is not necessarily trivial, and for other codimension-one splittings. It should be noticed that most of the constructions and the maps work in the topological category and the authors intend to examine if the splitting in the main theorem carries over to the topological category.

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2 Preliminaries

Let X be a finite dimensional smooth manifold and let I denote the unit interval [0, 1]. A *concordance* on X is a diffeomorphism

$$h : X \times I \to X \times I$$

so that $h|_{X \times \{i\}}$ is a diffeomorphism of $X \times \{i\}$ for $i = 0, 1$. If, in addition, h satisfies the condition that $h|X \times \{0\}$ is the identity map, then h is called a *pseudoisotopy*. If X has a boundary, we require that h be the identity on $X \times \{0\}$ and on $\partial X \times I$. An *isotopy* is a diffeomorphism h as above that preserves the I-coordinate. Let $\mathcal{P}(X)$ denote the group of isotopy classes of pseudoisotopies on X with multiplication induced by composition.

For the rest of the paper we assume that X is a compact finite dimensional smooth manifold. We write $\mathcal{P}_b(X\times\mathbb{R})$ for the subgroup of bounded, over $\mathbb{R}$, pseudoisotopies [18,20]. There is an isomorphism [1]:

$$\mathcal{P}_b(X\times\mathbb{R}) \cong Wh(X\times I),$$

where the Whitehead group classifies h-cobordisms over $X\times I$ which are the identity. Explicitly, the isomorphism is given by sending a pseudoisotopy $h \in \mathcal{P}_b(X\times\mathbb{R})$ to the h-cobordism:

$$h \mapsto (X\times[0,\infty)\times I) - h^{-1}(X\times(N,\infty)\times I),$$

where N was chosen so that $h^{-1}(X\times(N,\infty)\times I) \subset X\times[0,\infty)\times I$.

Let X be as above. The circle is written as $S^1 = [0,1]/\sim$, where we identity the two ends. Following [31], a pseudoisotopy h of $X\times S^1$ is called *relaxed* if it is isotopic to a concordance

$$h' : X\times S^1\times I \to X\times S^1\times I$$

so that $(X\times\{0\}\times I)\cap h'(X\times\{0\}\times I) = \emptyset$. We write $\mathcal{RP}(X\times S^1)$ for the subgroup of relaxed pseudoisotopies of $X\times S^1$.

3 Splitting the relaxed subgroup

There is an infinite transfer homomorphism:

$$\mathrm{tr}^\infty : \mathcal{P}(X\times S^1) \to \mathcal{P}_b(X\times\mathbb{R}^1).$$

More precisely, for $h \in \mathcal{P}(X\times S^1)$, $\mathrm{tr}^\infty(h)$ is the pull-back $\tilde{h}$ defined by:

$$\begin{tikzcd}
X\times\mathbb{R}\times I \arrow[r, "\tilde{h}"] \arrow[d, "E"] & X\times\mathbb{R}\times I \arrow[r, "\pi_2"] \arrow[d, "E"] & \mathbb{R} \arrow[d, "\exp"] \\
X\times S^1\times I \arrow[r, "h"] & X\times S^1\times I \arrow[r, "\pi_2"] & S^1
\end{tikzcd}$$

where $E(x,s,t) = (x,\exp(s),t)$.

Lemma 3.1 *With X as above, there is an exact sequence of groups*

$$0 \to \mathcal{P}(X\times I) \xrightarrow{i} \mathcal{RP}(X\times S^1) \xrightarrow{tr^\infty} \mathcal{P}_b(X\times\mathbb{R}^1) \to 0.$$

Proof For $f \in \mathcal{P}(X\times I)$, note that $f(x,s,0) = (x,s,0)$ and $f(x,j,t) = (x,j,t)$ for $x \in X$, $s,t \in I$, and $j = 0,1$. Define $i(f) \in \mathcal{P}(X\times S^1)$ to be the pseudoisotopy given by $i(f)(x,s,t) = f(x,s,t)$ with the identification $(x,0,t) \sim (x,1,t)$. Since $i(f)^{-1}(X\times\{0\}\times I) = X\times\{0\}\times I$, after an isotopy, $i(f)$ is relaxed.
Claim: i is one-to-one. If $f \in \mathcal{P}(X\times I)$ so that $i(f)$ is trivial, then there is an isotopy

$$F_t : X\times S^1\times I \to X\times S^1\times I$$

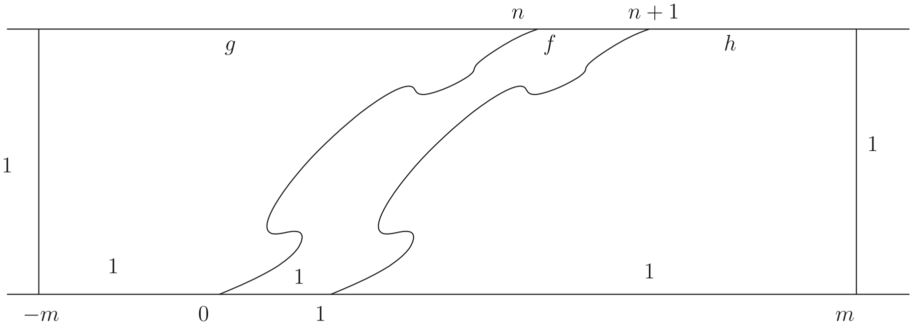
so that $F_0 = 1$ and $F_1 = i(f)$. Let $\overline{F}_t : X\times I\times I \to X\times S^1\times I$ be induced by F_t and lift $\overline{F}_t$ to $F'_t : X\times I\times I \to X\times\mathbb{R}\times I$ so that $E\circ F'_t = \overline{F}_t$. Note that F'_1 will agree with f after perhaps a shift in the $\mathbb{R}$ direction. By the Isotopy Extension Theorem, F'_t extends to an isotopy G_t on $X\times\mathbb{R}\times I$ with compact support. If G_t is the identity outside $X\times[-m,m]\times I$, we can consider G_t as an isotopy on $X \times [-m, m] \times I$. We may assume that G_t is the identity on $X \times \mathbb{R} \times \{0\}$.

Consider the restriction $G_1 : X \times [-m, 0] \times I \to X \times [-m, n] \times I$. After identifying $[-m, 0]$ and $[-m, n]$ with the interval I, this restriction of G_1 defines a pseudoisotopy $g \in \mathcal{P}(X \times I)$. Note that $g(x, 0, t) = G_1(x, -m, t) = (x, -m, t) \sim (x, 0, t)$ and $g(x, 1, t) = G_1(x, 0, t) = f(x, 0, t) = (x, 0, t)$.

Similarly, consider the restriction $G_1 : X \times [1, m] \times I \to X \times [n+1, m] \times I$. After identifying $[1, m]$ and $[n+1, m]$ with the interval I, this restriction of G_1 defines a pseudoisotopy $h \in \mathcal{P}(X \times I)$. Note that $h(x, 0, t) = G_1(x, 1, t) = f(x, 1, t) = (x, 1, t) \sim (x, 0, t)$ and $h(x, 1, t) = G_1(x, m, t) = (x, m, t) \sim (x, 1, t)$.

We consider an alternate product on $\mathcal{P}(X \times I)$ defined by $f \odot f'(x, s, t) = (x', s', t')$ where

$$\begin{cases} (x', 2s', t') = f(x, 2s, t) & \text{if } 0 \le s \le \frac{1}{2} \\ (x', 2s' - 1, t') = f'(x, 2s - 1, t) & \text{if } \frac{1}{2} \le s \le 1. \end{cases}$$



It is easy to see that $f \odot f'$ is isotopic to composition. The isotopy $G_t : X \times [-m, m] \times I \to X \times [-m, m] \times I$, after rescaling, gives an isotopy of $g \odot f \odot h$ to the identity.

Define a new isotopy G'_t on $X \times \mathbb{R} \times I$ by

$$G'_t(x, s, t) = \begin{cases} G_t(x, s, t) & \text{if } -m \le s \le 0 \\ G_t(x, s-1, t) & \text{if } 1 \le s \le m \end{cases}$$

This provides an isotopy of $g \odot h$ to the identity. Thus, combining these two isotopies, we have shown that if $i(f)$ is trivial, then f is isotopic to the identity in $\mathcal{P}(X \times I)$. Thus, i is an injection.

Claim: tr^∞ is onto. Let $\zeta : X \times \mathbb{R} \times I \to X \times \mathbb{R} \times I$ be the deck transformation. For any $g \in \mathcal{P}_b(X \times \mathbb{R})$, note that $g\zeta g^{-1}\zeta^{-1} \in \mathcal{P}_b(X \times \mathbb{R})$ is trivial. Therefore, there is an bounded over $\mathbb{R}$ isotopy $G_t : X \times \mathbb{R} \times I \to X \times \mathbb{R} \times I$ so that $G_0 = 1$, $G_1 = g\zeta g^{-1}\zeta^{-1}$, and $G_t|_{X \times \mathbb{R} \times \{0\}} = 1$. Define

$$g' : X \times I \times I \to X \times \mathbb{R} \times I, \quad g'(x, s, t) = G_s^{-1} g(x, s, t).$$

Then

- $g'(x, 0, t) = G_0^{-1} g(x, 0, t) = g(x, 0, t)$ and
- $g'(x, 1, t) = G_1^{-1} g(x, 1, t) = (\zeta g \zeta^{-1} g^{-1}) g(\zeta(x, 0, t)) = \zeta g(x, 0, t)$.

Thus, we can identify the ends of $X \times [0, 1] \times I$ to get a new function

$$g'' : X \times S^1 \times I \to X \times S^1 \times I.$$

Then $g'' \in \mathcal{P}(X \times S^1)$ and $\mathrm{tr}^\infty(g'')$ is isotopic to g.
Claim: $tr^\infty \circ i = 0$. Let $f \in \mathcal{P}(X \times I)$ and let $g = \mathrm{tr}^\infty(i(f))$. Then the pseudoisotopy g is bounded over $\mathbb{R}$:

$$g : X \times \mathbb{R} \times I \to X \times \mathbb{R} \times I.$$

The construction of g implies that:

$$g(X \times [0, \infty) \times I) = X \times [0, \infty) \times I.$$

After an isotopy, we can assume that g is the identity on a small collar of $X \times \{0\} \times I$. Let $g' = g|_{X \times [0,\infty) \times I}$. Then there is an isotopy ([34], Lemma 9.10):

$$G_t^+ : X \times [0, \infty) \times I \to X \times [0, \infty) \times I, \quad 0 \le t \le 1$$

so that G_t^+ is an isotopy from g' to the identity that also fixes $X \times \{0\} \times I$. Similarly, there is an isotopy G_t^- from $g|_{X \times (-\infty,0] \times I}$ to the identity fixing $X \times \{0\} \times I$. Then the union $G_t^+ \cup_{X \times \{0\} \times I} G_t^-$ provides an isotopy from g to the identity. Therefore, $g = 0$ in $\mathcal{P}_b(X \times \mathbb{R})$.
Claim: $Ker(tr^\infty) \subset Im(i)$. Let $h \in \mathcal{RP}(X \times S^1)$ so that $tr^\infty(h) = \tilde{h}$ is trivial in $\mathcal{P}_b(X \times \mathbb{R})$. Since $\mathcal{P}_b(X \times \mathbb{R})$ can be identified with $Wh(X \times I)$, the h-cobordism formed by $W = (X \times [0, \infty) \times I) - \tilde{h}^{-1}(X \times (0, \infty) \times I)$ is trivial. Since h is relaxed, W can be embedded in $X \times S^1 \times I$. Thus, there is a diffeomorphism

$$H : X \times [0, 1] \times I \to W \subset X \times S^1 \times I.$$

We may assume that $H|_{X \times \{0\} \times I}$ is the identity. For some $\epsilon > 0$, we can extend H by the identity to a diffeomorphism we will also denote by H from $X \times [-\epsilon, 1 + \epsilon] \times I$ to a neighborhood of W where $\epsilon > 0$ is chosen so that

$$H(X \times \{-\epsilon\} \times I) \cap H(X \times \{1 + \epsilon\} \times I) = \emptyset.$$

Define an isotopy h_t on $X \times S^1 \times I$ by

$$h_t(H(x, s, r)) = \begin{cases} (x, s, r) & \text{if } s \le -\epsilon \\ H\left(x, (1-t)s + t\left[-\epsilon + \dfrac{s+\epsilon}{1+\epsilon} \cdot \epsilon\right], r\right) & \text{if } -\epsilon < s \le 1 \\ H\left(x, (1-t)s + t\left[1 + \epsilon - \dfrac{1+\epsilon-s}{\epsilon} \cdot (1+\epsilon)\right], r\right) & \text{if } 1 < s \le 1+\epsilon \end{cases}$$

Note that h_t is well-defined and that h_0 is the identity. Also, $h_1 H(x, 1, r) = H(x, 0, r) = (x, 0, r)$ so h_1 takes $\tilde{h}^{-1}(X \times \{0\} \times I)$ to $X \times \{0\} \times I$. Let $h' = h \circ h_1$ so that h' is isotopic to h and $h'^{-1}(X \times \{0\} \times I) = X \times \{0\} \times I$. Define $j : X \times I \times I \to X \times S^1 \times I$ identifying $X \times \{0\} \times I$ to $X \times \{1\} \times I$. Consider the function $f : X \times I \times I \to X \times I \times I$ defined by

$$f(x, s, t) = \begin{cases} h' \circ j(x, s, t) & \text{if } 0 < s < 1, \\ (x, 0, t) & \text{if } s = 0, \\ (x, 1, t) & \text{if } s = 1. \end{cases}$$

Then f is a pseudoisotopy in $\mathcal{P}(X \times I)$ with $i(f) = h'$, which is isotopic to h. □

4 The basic construction

Let $h \in \mathcal{P}(X\times S^1)$. Lift h to the infinite cyclic cover:

$$\begin{array}{ccccc} X\times\mathbb{R}\times I & \xrightarrow{\tilde{h}} & X\times\mathbb{R}\times I & \xrightarrow{\pi_2} & \mathbb{R} \\ \downarrow E & & \downarrow E & & \downarrow \exp \\ X\times S^1\times I & \xrightarrow{h} & X\times S^1\times I & \xrightarrow{\pi_2} & S^1 \end{array}$$

where $E(x, s, t) = (x, \exp(s), t)$ and π_2 denotes projection onto the second factor.

Let $Y = \tilde{h}^{-1}(X\times\{0\}\times I)$. After an isotopy, we can assume that:

$$Y \subseteq X\times(0, N)\times I \quad \text{for some } N \in \mathbb{N}.$$

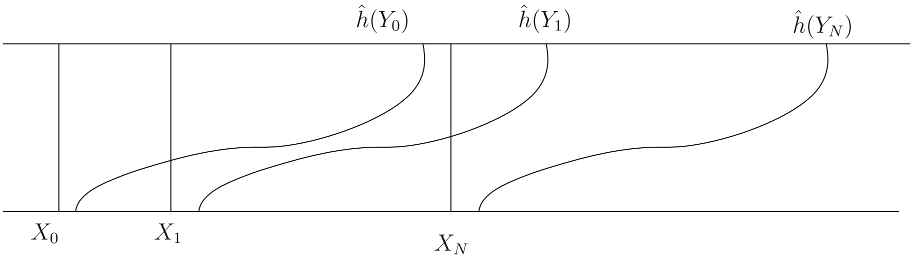
The pseudoisotopy $\tilde{h}$ induces a diffeomorphism:

$$\hat{h} : Y\times\mathbb{R} \to X\times\mathbb{R}\times I,$$

such that

(1) $\hat{h}(Y\times\{0\}) = Y$,
(2) $\hat{h}^{-1}|_{X\times\mathbb{R}\times\{0\}}$ is bounded over $\mathbb{R}$ isotopic to $\mathrm{id}_{X\times\mathbb{R}\times\{0\}}$.

Let $\zeta : X\times\mathbb{R}\times I \to X\times\mathbb{R}\times I$ be the covering transformation corresponding to translation by $+1$ on $\mathbb{R}$. Denote $X_i = X\times\{i\}\times I$ and $Y_i = Y\times\{i\}$, for $i \in \mathbb{R}$.



Also, let

$$\begin{aligned} A_N &= X\times[N, \infty)\times I - \hat{h}(Y\times(N, \infty)) \\ A''_N &= X\times[1, \infty)\times I - \hat{h}(Y\times(N, \infty)) \\ A_1 &= X\times[1, \infty)\times I - \hat{h}(Y\times(1, \infty)) \end{aligned}$$

Lemma 4.1 *Using the above notation:*

(1) *There is an isotopy*

$$a_t : A''_N \to A''_N$$

from the identity to a diffeomorphism

$$a_1 : A''_N \to A_N.$$

(2) *There is an isotopy*

$$a'_t : A''_N \to A''_N$$

from the identity to a diffeomorphism

$$a'_1 : A''_N \to A_1.$$

Proof For (1), choose $\varepsilon > 0$ such that $X \times [N, N+\varepsilon] \times I \subset A_N$ and define an isotopy $a_t : A''_N \to A''_N$ for $t \in [0,1]$ by

$$a_t(x,r,s) = \begin{cases} \left(x, (1-t)r + t\left[N+\varepsilon - \dfrac{N+\varepsilon-r}{N+\varepsilon-1}\varepsilon\right], s\right), & \text{if } 1 \le r \le N+\varepsilon \\ (x,r,s), & \text{if } r > N+\varepsilon. \end{cases}$$

Note that $a_0(x,t,s) = (x,r,s)$ for all $(x,r,s) \in A''_N$. Also,

$$a_t(x, N+\varepsilon, s) = (x, (1-t)(N+\varepsilon) + t(N+\varepsilon-0), s) = (x, N+\varepsilon, s).$$

So a_t is well-defined. For $t = 1$,

$$a_1(x,1,s) = (x,N,s).$$

But for $1 \le r \le N+\varepsilon$,

$$N \le N+\varepsilon - \frac{N+\varepsilon-r}{N+\varepsilon-1}\varepsilon \le N+\varepsilon.$$

Thus a_1 is a diffeomorphism from A''_N to A_N which is isotopic to the identity.

For (2), choose $\varepsilon' > 0$ such that $\hat{h}(Y \times [1-\varepsilon', 1]) \subset A_1$ and define an isotopy $a_t : A''_N \to A''_N$ for $t \in [0,1]$ by

$$a'_t\hat{h}(y,r) = \begin{cases} \hat{h}\left(y, (1-\varepsilon)r + t\left[1-\varepsilon + \dfrac{r-1+\varepsilon}{N-1+\varepsilon}\varepsilon\right]\right), & \text{if } r \le 1-\varepsilon' \\ \hat{h}(y,r), & \text{if } r < 1-\varepsilon'. \end{cases}$$

Note that $a'_0\hat{h}(y,r) = \hat{h}(y,r)$ for all (y,r). Also,

$$a'_t\hat{h}(y, 1-\varepsilon) = \hat{h}(y, (1-t)(1-\varepsilon) + t(1-\varepsilon+0)) = \hat{h}(y, 1-\varepsilon).$$

Thus, a'_t is well-defined. For $t = 1$,

$$a'_1\hat{h}(y,N) = \hat{h}(y,1).$$

But for $1-\varepsilon \le r \le N$,

$$1-\varepsilon \le 1-\varepsilon + \frac{r-1+\varepsilon}{k-1+\varepsilon}\varepsilon \le 1.$$

Thus a'_1 is a diffeomorphism from A''_N to A_1 which is isotopic to the identity. □

Corollary 4.2 *With the above notation, there is a diffeomorphism*

$$a : A_N \to A_1$$

isotopic to the identity.

Proof Set $a = a'_1 a_1^{-1}$. The result follows from Lemma 4.1. □

Using the above construction, we get the following:

Proposition 4.3 *There are two homomorphisms*

$$n, r : \mathcal{P}(X \times S^1) \to \mathcal{P}(X \times S^1).$$

so that for any $h \in \mathcal{P}(X \times S^1)$,

(1) $r(h) \in \mathcal{R}(X \times S^1)$, *and*
(2) $r(h) = h(n(h))^{-1}$.

Moreover, if $h \in \mathcal{R}(X \times S^1)$, $n(h) = 0$ *and* $r(h) = h$.

Proof Define

$$\overline{X} = X \times [0, N] \times I/(x, 0, s) \sim (x, N, s)$$

and a diffeomorphism:

$$f_1 : \overline{X} \to X \times S^1 \times I, \quad f_1(x, r, s) = E(x, r/N, s).$$

Similarly, define $\overline{Y} = Y \times [0, N]/(y, 0) \sim (y, N)$ and a diffeomorphism:

$$f_2 : \overline{Y} \to X \times S^1 \times I, \quad f_2(y, s) = E\hat{h}(y, s/N).$$

Also, define a diffeomorphism $H : \overline{Y} \to \overline{X}$ by

$$H(y, s) = \begin{cases} \hat{h}(y, s), & \text{if } \hat{h}(y, s) \in X \times [0, N] \times I, \\ \zeta^{-1} a\hat{h}(y, s), & \text{if } \hat{h}(y, s) \in A_N. \end{cases}$$

On $Y \times \{0\} \sim Y \times \{N\}$,

$$\begin{aligned} H(y, 0) &= \hat{h}(y, 0) \\ H(y, N) &= \zeta^{-1} a\hat{h}(y, N) = \zeta^{-1} a\hat{h}(y, 1) = \hat{h}(y, 0) \end{aligned}$$

On $X \times \{0\} \times I \sim X \times \{N\} \times I$, if $\hat{h}(y, s) = (x, N, t)$ then

$$H(y, s) = \begin{cases} (x, N, t), & \text{if } \hat{h}(y, s) \in X \times [0, N] \times I, \\ \zeta^{-1} a(x, N, t) = \zeta^{-1}(x, 1, t) = (x, 0, t), & \text{if } \hat{h}(y, s) \in A_N. \end{cases}$$

Thus H is well-defined.

Consider the map

$$f_1 H f_2^{-1} : X \times S^1 \times I \to X \times S^1 \times I$$

defined by the composition:

$$X \times S^1 \times I \xleftarrow{f_2} \overline{Y} \xrightarrow{H} \overline{X} \xrightarrow{f_1} X \times S^1 \times I.$$

There is an isotopy from $f_1 H f_2^{-1}$ to a map f' such that $f'|_{X \times S^1 \times \{0\}} = \mathrm{id}_{X \times S^1 \times \{0\}}$. Define $n(h) = f' \in \mathcal{P}(X \times S^1)$. We must show that $n(h)$ is well-defined.

Claim 1: $n(h)$ is independent of the choice of N. If N' were chosen rather than N, without loss of generality we may assume that $N' > N$. Let $N_s = (1 - s)N + sN'$, $0 \leq s \leq 1$. Let

$$\begin{aligned} A_{N_s} &= X \times [N_s, \infty) \times I - \hat{h}(Y \times (N_s, \infty)) \\ A''_{N_s} &= X \times [1, \infty) \times I - \hat{h}(Y \times (N_s, \infty)) \end{aligned}$$

For each $s \in I$, choose $\varepsilon_s > 0$ so that $X \times [N_s, N_s + \varepsilon_s] \times I \subseteq A_{N_s}$ and let $\varepsilon = \min_{s \in I}\{\varepsilon_s\}$. As above, define isotopies

$$a^s_t : A''_{N_s} \to A''_{N_s}, \quad a^s_0 = \mathrm{id}_{A''_{N_s}}, \quad a^s_1 : A''_{N_s} \xrightarrow{\cong} A_{N_s}.$$

Similarly choose $\varepsilon' > 0$ such that $\hat{h}(Y\times[1-\varepsilon', 1]) \subseteq A_1$ and define isotopies

$$a_t^{\prime s} : A''_{N_s} \to A''_{N_s}, \quad a_0^{\prime s} = \mathrm{id}_{A''_{N_s}}, \quad a_1^{\prime s} : A''_{N_s} \xrightarrow{\cong} A_1.$$

Let $a^s = a_1^{\prime s}(a_1^s)^{-1}$. Define $\overline{X}_s = X\times[0, N_s]\times I/(x, 0, t) \sim (x, N_s, t)$ and a diffeomorphism:

$$f_1^s : \overline{X}_s \to X\times S^1\times I.$$

Define $\overline{Y}_s = Y\times[0, N_s]/(y, 0) \sim (y, N_s)$ and a diffeomorphism:

$$f_2^s : \overline{Y}_s \to X\times S^1\times I.$$

Also define a diffeomorphism $H_s : \overline{Y}_s \to \overline{X}_s$ by

$$H_s(y, s) = \begin{cases} \hat{h}(y, s), & \text{if } \hat{h}(y, s) \in X\times[0, N_s]\times I, \\ \zeta^{-1}a^s\hat{h}(y, s), & \text{if } \hat{h}(y, r) \in A_{N_s}. \end{cases}$$

Then $f_1^s H_s(f_2^s)^{-1}$ defines an isotopy from the $n(h)$ as defined for N to the $n(h)$ as defined for N'.
Claim 2: $n(h)$ is independent of the choice of $\hat{h}$. If $\hat{h}'$ were chosen, rather than $\hat{h}$, note that $\hat{h}' = \zeta^r\hat{h}$ for some $r \in \mathbb{Z}$. Thus $Y'_N = Y_{r+N}$ and independence follows as above.
Claim 3: If h is relaxed, then $n(h)$ is trivial. If h is relaxed, then after possibly an isotopy,

$$\hat{h}(Y\times\{0\}) \subseteq X\times(0, 1)\times I$$

Thus $A_N = A''_N = A_1$ and $a = 1$. Note $\overline{X} = X\times S^1\times I$ and $f_1 = \mathrm{id}_{X\times S^1\times I}$. Also, $\overline{Y} = Y\times[0, 1]/\sim$ and the two ends are identified using the identity. Furthermore,

$$H(y, s) = \begin{cases} \hat{h}(y, s), & \text{if } \hat{h}(y, s) \in X\times[0, 1]\times I \\ \zeta^{-1}\hat{h}(y, s), & \text{if } \hat{h}(y, s) \in A_1 \end{cases}$$

Thus $n(h) = f_1 H f_2 = \mathrm{id}_{X\times S^1\times I}$.

Define $r(h) = h(n(h))^{-1}$, which is well-defined by the arguments above. Note that

$$r(h)^{-1}(X\times\{0\}\times I) \cap (X\times\{0\}\times I) = \emptyset$$

and therefore $r(h)$ is relaxed. Furthermore, if h is relaxed, then since $n(h)$ is trivial, it follows that $r(h)$ is isotopic to h. From [31], the maps n and r are homomorphisms. □

Remark 4.4

(1) Note that $n(h)$ essentially measures the difference between the initial pseudoisotopy h and the transfer of h that does not wrap around the circle completely (compare with Remark 8.2).
(2) The construction of r corresponds to the twist-gluing construction given in [23,35].

Let $\mathcal{NP}(X)$ be the image of n. The following is an immediate consequence of Proposition 4.3:

Theorem 4.5 *Let X be a closed finite-dimensional smooth manifold.*

(1) The inclusion map (forget-control map)

$$\phi : \mathcal{RP}(X\times S^1) \to \mathcal{P}(X\times S^1)$$

is a split injection with left inverse r.

(2) *There is an isomorphism of abelian groups:*

$$\mathcal{P}(X\times S^1) \xrightarrow{r+n} \mathcal{RP}(X\times S^1) + \mathcal{NP}(X).$$

5 The transfer map and relaxation

Let $\ell \in \mathbb{N}$ and $\times\ell : S^1 \to S^1$ denote the ℓ-fold transfer map. Consider the pull-back:

$$\begin{tikzcd}
X\times S^1\times I \arrow[r, "tr^\ell(h)"] \arrow[d, "1\times\ell\times 1"'] & X\times S^1\times I \arrow[r, "\pi_2"] \arrow[d, "1\times\ell\times 1"'] & S^1 \arrow[d, "\times\ell"] \\
X\times S^1\times I \arrow[r, "h"] & X\times S^1\times I \arrow[r, "\pi_2"] & S^1
\end{tikzcd}$$

We wish to show that the transfer map commutes with relaxation.

Proposition 5.1 *Let X be a closed manifold.*

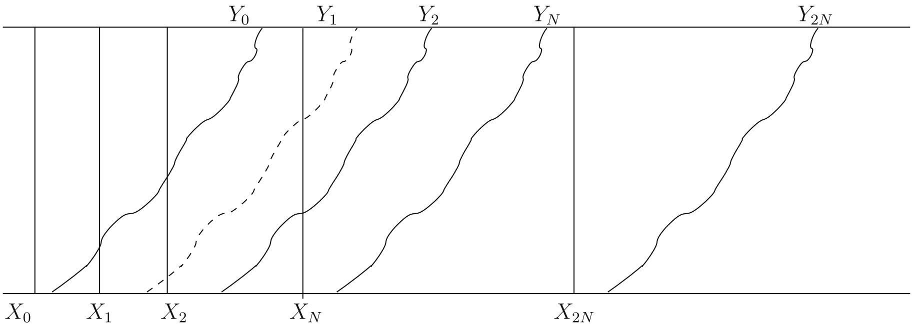
(1) *The transfer map commutes with the relaxation map:*

$$tr^\ell \circ r = r \circ tr^\ell, \quad \textit{for all} \ \ \ell \in \mathbb{N}.$$

(2) *Given $h \in \mathcal{P}(X\times S^1)$, there is an $N \in \mathbb{N}$ such that $tr^k(h) \in \mathcal{RP}(X\times S^1)$, for all $k \geq N$.*

Proof As above, let $Y = \tilde{h}^{-1}(X \times \{0\} \times I)$, with diffeomorphism $\hat{h} : Y \times \mathbb{R} \to X \times \mathbb{R} \times I$, $X_i = X \times \{i\} \times I \subset X \times \mathbb{R} \times I$, $Y_i = \hat{h}(Y \times \{i\})$, and ζ denotes the covering transformation of the infinite cyclic cover. Let $N \in \mathbb{N}$ so that $Y_0 \subset (X \times (0, N) \times I)$. To define $r(tr^\ell(h))$, set

$$\begin{aligned}
A_{\ell N} &= X\times[\ell N, \infty)\times I - \hat{h}(Y\times(\ell N, \infty))\\
A''_{\ell N} &= X\times[\ell, \infty)\times I - \hat{h}(Y\times(\ell N, \infty))\\
A_{\ell} \ \ &= X\times[\ell, \infty)\times I - \hat{h}(Y\times(\ell, \infty))
\end{aligned}$$



As above, we can construct a diffeomorphism $a' : A_{\ell N} \to A_\ell$ isotopic to the identity. Set

$$\begin{aligned}
\overline{X}' &= X\times[0, \ell N]\times I/(x, 0, t) \sim (x, \ell N, t)\\
\overline{Y}' &= Y\times[0, \ell N]/(x, 0) \sim (x, \ell N).
\end{aligned}$$

Again, we have diffeomorphisms:

$$\begin{aligned}
f'_1 &: \overline{X}' \xrightarrow{\cong} X\times S^1\times I\\
f'_2 &: \overline{Y}' \xrightarrow{\cong} X\times S^1\times I
\end{aligned}$$

Define a diffeomorphism $H' : \overline{Y}' \to \overline{X}'$ by

$$H'(y,s) = \begin{cases} \hat{h}(y,s), & \text{if } \hat{h}(y,s) \in X\times[0,\ell N]\times I, \\ \zeta^{-\ell}a'\hat{h}(y,s), & \text{if } \hat{h}(y,s) \in A_{\ell N}. \end{cases}$$

Then $r(tr^{\ell}(h))$ is defined by:

$$r(tr^{\ell}(h)) = hf_2'(H')^{-1}(f_1')^{-1}, \quad n(tr^{\ell}(h)) = f_1'H'(f_2')^{-1}.$$

Next, consider $tr^{\ell}(r(h))$. Let $\overline{X}$, $\overline{Y}$, f_1, and f_2 be defined as in the construction of $r(h)$. We generalize the diffeomorphism $a : A_N \to A_1$ to construct diffeomorphisms $a_i = \zeta^{(i-1)N}a\zeta^{-(i-1)N} : A_{iN} \to A_{i(N-1)+1}$. Then, define $\overline{H} : \overline{Y}' \to \overline{X}'$ by

$$\overline{H}(y,s) = \begin{cases} \hat{h}(y,s), & \text{if } \hat{h}(y,s) \in X\times[0,N]\times I \\ \zeta^{N}(\zeta^{-1}a\hat{h}(y,s)), & \text{if } \hat{h}(y,s) \in A_N \\ \hat{h}(y,s), & \text{if } \hat{h}(y,s) \in (X\times[N,2N]\times I) - A_N \\ \zeta^{N}(\zeta^{-1}a_2\hat{h}(y,s)), & \text{if } \hat{h}(y,s) \in A_{2N} \\ \dots & \\ \hat{h}(y,s), & \text{if } \hat{h}(y,s) \in (X\times[(\ell-1)N,\ell N]\times I) - A_{(\ell-1)N} \\ \zeta^{(\ell-1)N}(\zeta^{-1}a_{\ell}\hat{h}(y,s)), & \text{if } \hat{h}(y,s) \in A_{\ell N} \end{cases}$$

Then the following diagrams commute:

$$\begin{array}{ccc} \overline{Y}' & \xrightarrow{\overline{H}} & \overline{X}' \\ \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle 1\times\ell\times 1} \\ \overline{Y} & \xrightarrow{H} & \overline{X}, \end{array} \qquad \begin{array}{ccc} \overline{X}' & \xrightarrow{f_1'} & X\times S^1\times I \\ \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle 1\times\ell\times 1} \\ \overline{X} & \xrightarrow{f_1} & X\times S^1\times I, \end{array} \qquad \begin{array}{ccc} \overline{Y}' & \xrightarrow{f_2'} & X\times S^1\times I \\ \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle 1\times\ell\times 1} \\ \overline{Y} & \xrightarrow{f_2} & X\times S^1\times I, \end{array}$$

Thus, we have

$$\begin{array}{rcccccc} tr^{\ell}(r(h)) : X\times S^1\times I & \xrightarrow{(f_2')^{-1}} & \overline{Y}' & \xrightarrow{\overline{H}} & \overline{X}' & \xrightarrow{f_1'} & X\times S^1\times I \\ \downarrow{\scriptstyle 1\times\ell\times 1} & & \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle \times\ell} & & \downarrow{\scriptstyle 1\times\ell\times 1} \\ r(h) : X\times S^1\times I & \xrightarrow{(f_2)^{-1}} & \overline{Y} & \xrightarrow{H} & \overline{X} & \xrightarrow{f_1} & X\times S^1\times I. \end{array}$$

Since the map a is isotopic to the identity, it follows that $\overline{H}$ and H' are isotopic, so for any $\ell \in \mathbb{N}$,

$$tr^{\ell}(r(h)) = r(tr^{\ell}(h)).$$

For $N \in \mathbb{N}$ chosen so that $Y \subset (X \times [0,N] \times I)$, it is clear that $tr^N(h)$ is relaxed. Thus, for any $k \geq N$, $tr^k(h)$ is also relaxed. □

Recall from Theorem 4.5 that every element of $\mathcal{P}(X\times S^1)$ is the sum of a relaxed element and an element from $\mathcal{NP}(X)$. Thus we have the following.

Corollary 5.2 *Let X be a closed manifold.*

(1) Given $h \in \mathcal{NP}(X)$, there is an $N \in \mathbb{N}$ such that $tr^k(h) = 0$ for all $k \geq N$.
(2) $(\mathcal{RP}(X\times S^1)) \cap \mathcal{NP}(X) = \{0\}$.

Remark 5.3 The result on transfers for the Whitehead groups is contained in [23] (also [24,32], and [33] for a more algebraic approach). It states that the controlled part is preserved under the transfers and the Nil-parts vanish after finitely many transfers.

Therefore we have the following result.

Theorem 5.4 *Let X be a closed finite dimensional smooth manifold. Then*

$$\mathcal{P}(X\times S^1) \cong \mathcal{RP}(X\times S^1)\oplus\mathcal{NP}(X).$$

given by $h = r(h) + n(h)$. Furthermore

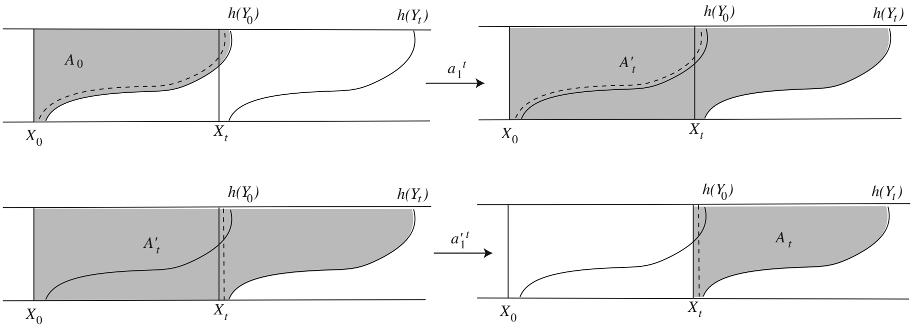
$$\mathcal{P}(X\times S^1) \cong \mathcal{P}(X\times I)\oplus\mathcal{P}_b(X\times\mathbb{R})\oplus\mathcal{NP}(X).$$

6 A splitting of $\mathcal{NP}(X)$

By [16], we know that $\mathcal{NP}(X)$ ought to split into two dual components, corresponding to the two ends of $\mathbb{R}$. We give an explicit construction of this splitting, at the cost of crossing with another I-factor.

For $0 \le t \le 1$, let

$$\begin{aligned} A_t &= X\times[t,\infty)\times I - \hat{h}(Y\times(t,\infty)) \\ A_0 &= X\times[0,\infty)\times I - \hat{h}(Y\times(0,\infty)) \\ A'_t &= X\times[0,\infty)\times I - \hat{h}(Y\times(t,\infty)) \end{aligned}$$



Lemma 6.1 *Let $A = A_0 \times I/(z, 0) \sim (\alpha_1^{-1}\zeta(z), 1)$. There is a diffeomorphism $\alpha : A \to X \times S^1\times I \times I$.*

Proof We will first define a diffeomorphism $\alpha_t : A_0 \to A_t$ isotopic to the identity on $X \times \mathbb{R} \times I$. Choose $\varepsilon > 0$ such that $\tilde{h}(Y \times [-\varepsilon, 0]) \subset A_0$ and for $T \in [0, 1]$, define

$$a^t_T(\tilde{h}(y,r)) = \begin{cases} \tilde{h}\left(y, (1-T)r + T\left(-\varepsilon + \dfrac{t+\varepsilon}{\varepsilon}(r+\varepsilon)\right)\right), & \text{if } -\varepsilon \le r \le 0 \\ \tilde{h}(y,r), & \text{if } r < -\varepsilon. \end{cases}$$

Note that a^t_T is well-defined, $a^t_0 = \mathrm{id}$, and $a^t_1(A_0) = A'_t$. Moreover, a^t_T can be extended to an isotopy on $X \times \mathbb{R} \times I$ so that $a^t_0 = \mathrm{id}$.

Choose $\varepsilon' > 0$ so that $X \times [t, t+\varepsilon'] \times I \subset A_t$ and for $T \in [0, 1]$ define

$$a'^t_T(x,r,s) = \begin{cases} \left(x, (1-T)r + T\left(t + \dfrac{\varepsilon'}{t+\varepsilon'}\cdot r\right), s\right), & \text{if } 0 \le r \le t+\varepsilon' \\ (x,r,s), & \text{if } r > t+\varepsilon'. \end{cases}$$

Again, a''^t_T is well-defined, $a''^t_0 = \mathrm{id}$, and $a''^t_1(A'_t) = A_t$. Also, a''^t_T can be extended to an isotopy on $X \times \mathbb{R} \times I$ so that $a''^t_0 = \mathrm{id}$. Define $\alpha_t : A_0 \to A_t$ by $\alpha_t = a''^t_1 a^t_1$ and note that $\alpha_0 = \mathrm{id}$. Let $A = A_0 \times I/(z, 0) \sim (\alpha_1^{-1}\zeta(z), 1)$ and define $\alpha : A \to X \times S^1 \times I \times I$ by

$$\alpha(z,t) = \left(E\alpha_t(z), \frac{\pi\alpha_t(z) - t}{\pi\alpha_t(z) - \pi'\tilde{h}^{-1}\alpha_t(z)}\right)$$

Note that

$$\alpha(z,0) = \left(E(z), \frac{\pi(z)}{\pi(z) - \pi'\tilde{h}^{-1}(z)}\right)$$

$$\begin{aligned}\alpha(\alpha_1^{-1}\zeta(z), 1) &= \left(E\alpha_1\alpha_1^{-1}\zeta(z), \frac{\pi\alpha_1\alpha_1^{-1}\zeta(z) - 1}{\pi\alpha_1\alpha_1^{-1}\zeta(z) - \pi'\tilde{h}^{-1}\alpha_1\alpha_1^{-1}\zeta(z)}\right)\\ &= \left(E\zeta(z), \frac{\pi\zeta(z) - 1}{\pi\zeta(z) - \pi'\tilde{h}\zeta(z)}\right)\\ &= \left(E(z), \frac{\pi(z)}{\pi(z) - \pi'\tilde{h}(z)}\right) = \alpha(z,0).\end{aligned}$$

Since $E(\zeta(z)) = E(z)$, $\pi(\zeta(z)) = \pi(z) + 1$, and $\pi'\tilde{h}^{-1}\zeta(z) = \pi'\tilde{h}^{-1}(z) + 1$, α is well-defined.

If $\alpha(z,t) = \alpha(z',t')$, then $E\alpha_t(z) = E\alpha_{t'}(z')$ and so $\alpha_{t'}(z') = \zeta^n\alpha_t(z)$ for some $n \in \mathbb{Z}$. Furthermore,

$$\begin{aligned}\frac{\pi\alpha_t(z) - t}{\pi\alpha_t(z) - \pi'\tilde{h}^{-1}\alpha_t(z)} &= \frac{\pi\alpha_{t'}(z') - t'}{\pi\alpha_{t'}(z') - \pi'\tilde{h}^{-1}\alpha_{t'}(z')}\\ &= \frac{\pi\zeta^n\alpha_t(z) - t'}{\pi\zeta^n\alpha_t(z) - \pi'\tilde{h}^{-1}\zeta^n\alpha_t(z)}\\ &= \frac{\pi\alpha_t(z) + n - t'}{\pi\alpha_t(z) + n - \pi'\tilde{h}^{-1}\alpha_t(z) - n}\\ &= \frac{\pi\alpha_t(z) + n - t'}{\pi\alpha_t(z) - \pi'\tilde{h}^{-1}\alpha_t(z)}\end{aligned}$$

Thus $n = t' - t$. Since $0 \le t, t' \le 1$, it follows that either $t = t'$ or $t = 0$ and $t' = 1$. If $t = t'$, then $\alpha_{t'}(z') = \alpha_t(z') = \zeta^0\alpha_t(z) = \alpha_t(z)$ and so $z = z'$. If $t = 0$ and $t' = 1$, then $\alpha_1(z') = T\alpha_0(z) = \zeta(z)$ and so $z' = \alpha_1\zeta(z)$, so $(z', t') \sim (z, t)$ in A. In both cases, it follows that α is a diffeomorphism. Note that α and A are not independent of the choice of $\tilde{h}$, but if $\tilde{h}'$ were chosen instead then $\tilde{h}' = \zeta^n\tilde{h}$ for some $n \in \mathbb{Z}$.

Also note that since for $z \in A_0$, $\alpha_t(z)$ lies in A_t, it follows that

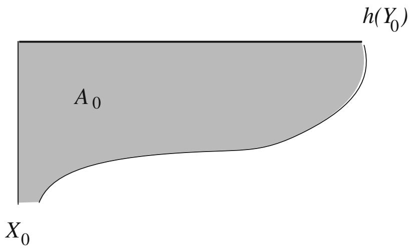
$$\pi\alpha_t(z) \ge t \ge \pi'\tilde{h}^{-1}\alpha_t(z).$$

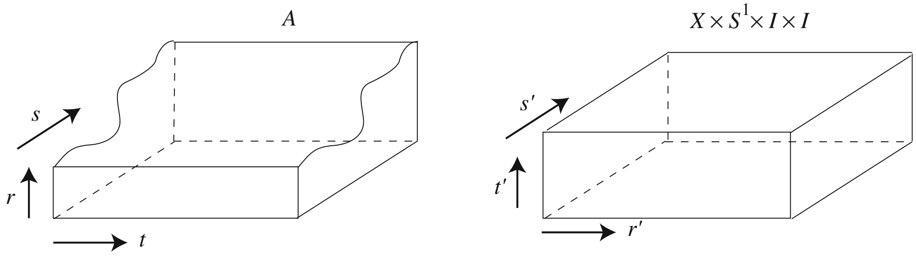
Thus,

$$0 \le \frac{\pi\alpha_t(z) - t}{\pi\alpha_t(z) - \pi'\tilde{h}^{-1}\alpha_t(z)} \le 1.$$

This quantity is 0 when $z \in X \times \{0\} \times I$ and it is 1 when $z \in \tilde{h}(Y \times \{0\})$. □

Points in $A = A_0 \times I/\sim$ are denoted $((x, r, s), t)$ where $(x, r, s) \in A_0 \subset X \times \mathbb{R} \times I$. Points in $X \times S^1 \times I \times I$ are denoted (x', r', s', t').

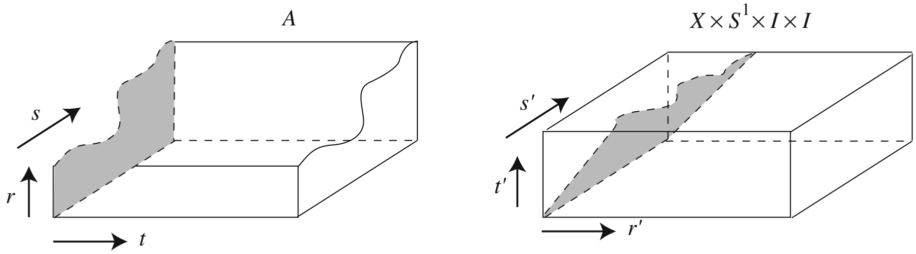




A few simple computations verify that α takes the floor and ceiling of A, respectively to the floor and ceiling of $X \times S^1 \times I \times I$, as pictured above. Similarly, α takes the front and back faces of A, respectively to the front and back faces of $X \times S^1 \times I \times I$. Finally, consider a point $((x, r, s), 0)$ on the left face of A, which is to be glued to the right face. Then,

$$\begin{aligned}
\alpha((x, r, s), 0) &= \left(E\alpha_0(x, r, s), \frac{\pi\alpha_0(x, r, s) - 0}{\pi\alpha_0(x, r, s) - \pi'\tilde{h}^{-1}\alpha_0(x, r, s)} \right) \\
&= \left(E(x, r, s), \frac{\pi(x, r, s)}{\pi(x, r, s) - \pi'\tilde{h}^{-1}(x, r, s)} \right) \\
&= \left(E(x, r, s), \frac{r}{r - \pi'\tilde{h}^{-1}(x, r, s)} \right)
\end{aligned}$$

This is drawn below for h relaxed:



We can adapt the construction above after scaling to obtain a relaxed space. Let

$$\begin{aligned}
A^t &= \left(1 \times \frac{1}{N} \times 1\right) \left[(X \times [t, \infty) \times I) - \tilde{h}(Y \times (Nt, \infty)) \right] \\
A^0 &= \left(1 \times \frac{1}{N} \times 1\right) \left[(X \times [0, \infty) \times I) - \tilde{h}(Y \times (0, \infty)) \right] \\
A'^t &= \left(1 \times \frac{1}{N} \times 1\right) \left[(X \times [0, \infty) \times I) - \tilde{h}(Y \times (Nt, \infty)) \right]
\end{aligned}$$

As above, we construct isotopies $\bar{a}_T^t$ and $\bar{a}_T'^t$ on $X \times \mathbb{R} \times I$ so that $\bar{a}_0^t = \bar{a}_0'^t = \mathrm{id}$, and $\bar{a}_1^t(A^0) = A'^t$ and $\bar{a}_1'^t(A'^t) = A^t$.

Define $\alpha_t' : A^0 \to A^t$ by $\alpha_t' = \bar{a}_1'^t \bar{a}_1^t$. Define $\gamma_0 : A_0 \to A^0$ by $\gamma_0 = (1 \times \frac{1}{N} \times 1)$ and define $\gamma_t : A_t \to A^t$ by $\gamma_t = \alpha_t' \gamma_0 \alpha_t^{-1}$. Let $A' = A^0 \times I/(z, 0) \sim (\alpha_1'^{-1} \zeta'(z), 1)$ where $\zeta' = \gamma_1 \zeta \gamma_0^{-1}$. Define $\alpha' : A' \to X \times S^1 \times I \times I$ by

$$\alpha'(z, t) = \left(E\alpha_t'(z), \frac{\pi\alpha_t'(z) - t}{\pi\alpha_t'(z) - \pi'\tilde{h}^{-1}\alpha_t'(z)} \right)$$

As above, α' is a diffeomorphism. Finally, note that $\gamma_0 \times I : A_0 \times I \to A^0 \times I$ induces a diffeomorphism $\gamma : A \to A'$.

Proposition 6.2 *For $h \in \mathcal{P}(X \times S^1)$ there is an element $n_+(h) \in \mathcal{P}(X \times S^1 \times I)$ with the following properties:*

(1) $n_+(h)|_{X \times S^1 \times I \times \{1\}}$ is isotopic to $n(h)$,
(2) $n_+(h) = 1$ if and only if h is relaxed, and
(3) For any h, there is a $N \in \mathbb{N}$ so that $n_+(tr^\ell(h)) = 1$ for all $\ell \geq N$.

Proof Define $n_+(h) : X \times S^1 \times I \times I \to X \times S^1 \times I \times I$ by

$$n_+(h) = \alpha' \gamma \alpha^{-1}.$$

This is a diffeomorphism since α, α', and γ are. Note that on $(Y \times \mathbb{R}) \cap (X \times \mathbb{R} \times \{0\})$, $\tilde{h}$ is isotopic to the identity. After this isotopy, we have $n_+(h)(x, r, s, 0) = (x, r, s, 0)$. Thus $n_+(h)$ is a pseudoisotopy. Tracing the image on $X \times S^1 \times I \times \{1\}$ shows that this is isotopic to $n(h)$ as defined previously.

If h happens to be relaxed, then we can choose $N = 1$. In this case, $A^t = A_t$ and $\alpha_t = \alpha_t'$. It follows that $\alpha = \alpha'$ and $\gamma = \mathrm{id}$. Thus, $n_+(h) = 1$ for h relaxed. From this, the comment in the previous paragraph, and Proposition 4.3, it follows that $n_+(h)$ is trivial if and only if h is relaxed, which is true if and only if $n(h)$ is trivial.

For X compact, there is a value of $N \in \mathbb{Z}$ such that $Y \subset X \times [0, N] \times I \subset X \times \mathbb{R} \times I$. For any $\ell \geq N$, the transfer map $tr^\ell(h)$ is relaxed. Therefore, for any $h \in \mathcal{P}(X \times S^1)$ there is a $N \in \mathbb{Z}$ so that for all $\ell \geq N$, $n_+(tr^\ell(h)) = 1$. ☐

Consider the suspension map $\Sigma : \mathcal{P}(X \times S^1) \to \mathcal{P}(X \times S^1 \times I)$ defined by $\Sigma(f) = f \times 1_I$. Extending the construction above to $n_+(h)$ easily gives us the following result:

Corollary 6.3 *For $h \in \mathcal{P}(X \times S^1)$, $n_+(n_+(h)) = \Sigma(n_+(h))$.*

The construction of $n_+(h)$ was based on the h-cobordism $A_0 = X \times [0, \infty) \times I - \hat{h}(Y \times (0, \infty))$. We could have chosen to work with the inverse of this cobordism instead, to construct a different (but dual) nilpotent pseudoisotopy:

Proposition 6.4 *For $h \in \mathcal{P}(X \times S^1)$ there is an element $n_-(h) \in \mathcal{P}(X \times S^1 \times I)$ such that*

(1) $n_-(h)|_{X \times S^1 \times I \times \{1\}}$ is isotopic to $n(h)$,
(2) $n_-(h) = 1$ if and only if h is relaxed, and
(3) For any h, there is a $N \in \mathbb{N}$ so that $n_+(tr^\ell(h)) = 1$ for all $\ell \geq N$.

Proof Let $N \in \mathbb{N}$ so $\tilde{h}(Y \times \{0\}) \subset (X \times (0, N) \times I)$ and let

$$B_t = (X \times (-\infty, t] \times I) - \hat{h}(Y \times (-\infty, -N + t))$$
$$B_0 = (X \times (-\infty, 0] \times I) - \hat{h}(Y \times (-\infty, -N))$$
$$B_t' = (X \times (-\infty, t] \times I) - \hat{h}(Y \times (-\infty, -N))$$

Choose $\varepsilon > 0$ so that $X \times [-\varepsilon, 0] \times I \subset B_0$ and $\varepsilon' > 0$ so that $\tilde{h}(Y \times [-N+t, -N+t+\varepsilon']) \subset B_t$. Define

$$b_T^t(x, r, s) = \begin{cases} \left(x, (1-T)r + T\left(-\varepsilon + \frac{t+\varepsilon}{\varepsilon}(r+\varepsilon)\right), s\right), & \text{if } -\varepsilon \le r \le 0 \\ (x, r, s), & \text{if } r < -\varepsilon. \end{cases}$$

and

$$b'^t_T \tilde{h}(y, r))$$
$$= \begin{cases} \tilde{h}\left(y, (1-T)r + T\left(-N + t + \frac{\varepsilon'}{t+\varepsilon'}(r+N)\right)\right), & \text{if } -N \le r \le -N+t+\varepsilon' \\ \tilde{h}(y, r), & \text{if } r > -N+t+\varepsilon'. \end{cases}$$

As above, $b_0^t = b'^t_0 = \text{id}$, these maps can be extended to isotopies on $X \times \mathbb{R} \times I$, and $b_1^t(B_0) = B_t'$ and $b'^t_1(B_t') = B_t$. Define β_t isotopic to the identity on $X \times \mathbb{R} \times I$ by $\beta_t = b'^t_1 b_1^t$ so $\beta_t(B_0) = B_t$. Define $B = B_0 \times I/(z, 0) \sim (\beta_1^{-1}\zeta(z), 1)$. As above, there is a diffeomorphism $\beta : B \to X \times S^1 \times I \times I$ defined by

$$\beta(z, t) = \left(E\beta_t(z), \frac{\pi\beta_t(z) - t}{\pi\beta_t(z) - \pi'\tilde{h}^{-1}\beta_t(z)}\right)$$

Let

$$\begin{aligned} B^t &= (1 \times \tfrac{1}{N} \times 1)[(X \times (-\infty, t] \times I) - \hat{h}(Y \times (-\infty, N(t-1)))] \\ B^0 &= (1 \times \tfrac{1}{N} \times 1)[(X \times (-\infty, 0] \times I) - \hat{h}(Y \times (-\infty, -N))] \\ B'^t &= (1 \times \tfrac{1}{N} \times 1)[(X \times (-\infty, t] \times I) - \hat{h}(Y \times (-\infty, -N))] \end{aligned}$$

Choose $\delta > 0$ so that $(1 \times \frac{1}{N} \times 1)(X \times [-\delta, 0] \times I) \subset B^0$ and $\delta' > 0$ so that $(1 \times \frac{1}{N} \times 1)\tilde{h}([N(t-1), N(t-1)+\delta']) \subset B^t$. Define

$$\bar{b}_T^t(x, r, s)$$
$$= \begin{cases} \left(1 \times \frac{1}{N} \times 1\right)\left(x, (1-T)r + T\left(-\delta + \frac{t+\delta}{\delta}(r+\delta)\right), s\right), & \text{if } -\delta \le r \le 0 \\ \left(1 \times \frac{1}{N} \times 1\right)(x, r, s), & \text{if } r < -\delta. \end{cases}$$

and

$$\bar{b}'^t_T \left(1 \times \frac{1}{N} \times 1\right)\tilde{h}(y, r))$$
$$= \begin{cases} \left(1 \times \frac{1}{N} \times 1\right)\tilde{h}(y, (1-T)r + T(N(t-1) + \frac{\delta'}{N + N(t-1) + \delta'}(r+N)), \\ \qquad \text{if } -N \le r \le N(t-1) + \delta' \\ \left(1 \times \frac{1}{N} \times 1\right)\tilde{h}(y, r), \qquad \text{if } r > N(t-1) + \delta'. \end{cases}$$

Again, $\bar{b}_0^t = \bar{b}'^t_0 = \text{id}$, these maps can be extended to isotopies on $X \times \mathbb{R} \times I$, and $\bar{b}_1^t(B^0) = B'^t$ and $\bar{b}'^t_1(B'^t) = B^t$. Define β_t' isotopic to the identity on $X \times \mathbb{R} \times I$ by $\beta_t' = \bar{b}'^t_1 \bar{b}_1^t$ so

$\beta_t'(B^0) = B^t$. Define $B' = B_0' \times I/(z,0) \sim (\beta'^{-1}_1\zeta'(z), 1)$. As above, there is a diffeomorphism $\beta' : B' \to X \times S^1 \times I \times I$ defined by

$$\beta'(z,t) = \left(E\beta_t'(z), \frac{\pi\beta_t'(z) - t}{\pi\beta_t'(z) - \pi'\tilde{h}^{-1}\beta_t'(z)} \right)$$

Define $\gamma_0' : B_0 \to B^0$ by $\gamma_0' = (1 \times \frac{1}{N} \times 1)$ and then define $\gamma_t' : B_t \to B^t$ by $\gamma_t' = \beta_t'\gamma_0'\beta_t^{-1}$. As above, $\gamma_0' \times 1$ extends to a diffeomorphism $\gamma' : B \to B'$.

Finally, define the pseudoisotopy

$$n_-(h) = \beta'\gamma'\beta^{-1} : X \times S^1 \times I \times I \to X \times S^1 \times I \times I$$

As above for $n_+(h)$, $n_-(h)|_{X\times S^1\times I\times\{0\}}$ is isotopic to the identity and $n_-(h)|_{X\times S^1\times I\times\{1\}}$ is isotopic to $n(h)$. If h is relaxed, $n_-(h)$ is trivial and so there exists $N \in \mathbb{Z}$ so that for all $\ell \geq N$, $n_-(tr^{\ell}(h)) = 1$. □

Remark 6.5 Notice that the construction of the maps $n_\pm$ is related to the classical construction of the classical projections to the Nil-groups [3,23,24,27,28,32,33]. The construction of n_+ looks at the ‘restriction’ of the pseudoisotopy to the positive end and it uses the analogue of mapping tori for the definition. The definition of n_- is similar but it uses the restriction to the negative end instead.

In general, for $f \in \mathcal{P}(Z \times I)$ with $f|_{Z\times I\times\{0\}} = 1$ and $f|_{Z\times\{i\}\times I} = 1$ for $i = 0, 1$, let $f_1 = f|_{Z\times I\times\{1\}}$ and note that $f_1 \in \mathcal{P}(Z)$. The suspension map

$$\Sigma : \mathcal{P}(Z) \to \mathcal{P}(Z\times I)$$

is defined as follows. The suspension of $g \in \mathcal{P}(Z)$ is defined to be $\Sigma g = g \times 1 \in \mathcal{P}(Z \times I)$ after an isotopy so that $\Sigma g = 1$ on $Z \times \{0, 1\} \times I$. Let $\psi : Z \times I \times I \to Z \times I \times I$ be defined by

$$\psi(z, s, t) = (z, s, 1 - t).$$

This involution defines the conjugate of $f \in \mathcal{P}(Z \times I)$ as

$$\overline{f} = (f_1^{-1} \times 1)\psi f \psi \in \mathcal{P}(Z \times I)$$

Recall Hatcher, Wagoner, and Igusa’s identification [14,15,29,30]

$$\mathcal{P}(Z) \cong Wh_2(\pi_1(Z)) \oplus Wh_1(\pi_1(Z) : \mathbb{Z}_2 \times \pi_2(Z)),$$

if dim ≥ 7. Note that in this setting, $[\overline{f}] = (-1)^n\overline{[f]}$

Lemma 6.6 *For $f \in \mathcal{P}(Z \times I)$,*

$$[\Sigma f_1] = [f] + \overline{[f]}.$$

Proof The argument is modified from Lemma 5.3 of Part II of [15]. Define another involution $\phi : Z \times I \times I \to Z \times I \times I$ by

$$\phi(z, s, t) = (z, 1 - s, t).$$

Define $\hat{f} = \phi\psi(f_1^{-1} \times 1)f\psi\phi$. Note that $\hat{f} \in \mathcal{P}(Z \times I)$ and $\hat{f}$ is isotopic to $(f_1^{-1} \times 1)f = \Sigma(f_1^{-1}) \circ f$ by a rotation in $I \times I$. Also note that $\hat{f}$ differs from $\overline{f}$ by the involution ϕ.

Since ϕ induces the identity on $\pi_*(Z)$, it also induces the identity on $\mathcal{P}(Z \times I)$ in the Hatcher–Wagoner–Igusa setting. Therefore, in this setting,

$$[\hat{f}] = [\overline{f}] = (-1)^n \overline{[f]}$$

and thus

$$[\Sigma(f_1)] = [f_1] = [f] - [\hat{f}] = [f] + (-1)^{n-1}\overline{[f]}.$$

□

Corollary 6.7 *In the Hatcher–Wagoner–Igusa setting, for any $h \in \mathcal{P}(X \times S^1)$,*

$$[\Sigma(n(h))] = [n(h)] = [n_+(h)] + (-1)^{n-1}\overline{[n_+(h)]}, \quad \textit{and}$$

$$[\Sigma(n(h))] = [n(h)] = [n_-(h)] + (-1)^{n-1}\overline{[n_-(h)]}.$$

Proposition 6.8 *For any $h \in \mathcal{P}(X \times S^1)$, $\overline{n_-(h)} \circ n_+(h)$ is trivial.*

Proof Define

$$B * A = B \cup A / X \times \{0\} \times I.$$

Note that for $(x, 0, s) \in X \times \{0\} \times I$, $\alpha_1^{-1}\zeta(x, 0, s) = (x, 0, s) = \beta_1^{-1}\zeta(x, 0, s)$, so $B * A$ is well-defined. Recall that $\beta : B \to X \times S^1 \times I \times I$ takes $(X \times \{0\} \times I) \times I \subset B_0 \times I$ to $X \times S^1 \times I \times \{0\}$ and $\tilde{h}(Y \times \{-N\}) \times I \subset B_0 \times I$ to $X \times S^1 \times I \times \{1\}$. Define $\overline{\beta} : B \to X \times S^1 \times I \times I$ by $\overline{\beta} = \psi \circ \beta$.

We define a diffeomorphism $\overline{\beta} * \alpha : B * A \to X \times S^1 \times I \times I$ (with a rescaling in the last I factor by $\frac{1}{2}$):

$$\overline{\beta} * \alpha(z) = \begin{cases} \left(x, r, u, s, \dfrac{t}{2}\right) & \text{if } z \in B \text{ and } \overline{\beta(z)} = (x, r, s, t) \\ \left(x, r, u, s, \dfrac{t+1}{2}\right) & \text{if } z \in A \text{ and } \alpha(z) = (x, r, s, t). \end{cases}$$

Since $\overline{\beta}$ and α agree on $(X \times \{0\} \times I) \times I \subset B \cup A$, $\overline{\beta} * \alpha$ is well-defined. Note that B_0 and A_0 are inverse cobordisms so there is a diffeomorphism of $B_0 \cup A_0/(X \times \{0\} \times I)$ to $X \times I \times I$. This induces a diffeomorphism

$$F : X \times S^1 \times I \times I \to B * A.$$

Then $(\overline{\beta} * \alpha)F : X \times S^1 \times I \times I \to X \times S^1 \times I \times I$ is a diffeomorphism that is the identity on $X \times S^1 \times I \times \{0\}$ and is isotopic to a rotation by $2\pi N$ on $X \times S^1 \times I \times \{1\}$.

We define $B' * A'$ and a diffeomorphism $\overline{\beta'} * \alpha' : B' * A' \to X \times S^1 \times I \times I$ similarly. Note that again there is a diffeomorphism $F' : X \times S^1 \times I \times I \to B' * A'$. Then

$$(\overline{\beta'} * \alpha')F' : X \times S^1 \times I \times I \to X \times S^1 \times I \times I$$

is a diffeomorphism that is the identity on $X \times S^1 \times I \times \{0\}$ and is isotopic to a rotation by 2π on $X \times S^1 \times I \times \{1\}$.

Finally, define $\overline{\gamma'} * \gamma : B * A \to B' * A'$ by

$$\overline{\gamma'} * \gamma(z) = \begin{cases} \overline{\gamma'}(z), & \text{if } z \in B \\ \gamma(z), & \text{if } z \in A. \end{cases}$$

Since $\overline{\gamma'}(x,0,y,s) = (1\times\frac{1}{N}\times 1\times 1)(x,0,y,s) = \gamma(x,0,y,s)$, $\overline{\gamma'}*\gamma$ is also well-defined. We can then define $\overline{n_-(h)} * n_+(h) : X\times S^1\times I\times I \to X\times S^1\times I\times I$ by

$$\overline{n_-(h)} * n_+(h) = (\overline{\beta'}*\alpha')(\overline{\gamma'}*\gamma)(\overline{\beta}*\alpha)^{-1}.$$

Note that for pseudoisotopies the product $*$ thus defined is isotopic to composition by [15]. Furthermore,

$$\overline{n_-(h)} * n_+(h) = F'\circ\left(1\times\frac{1}{N}\times 1\times 1\times 1\right)\circ F^{-1}.$$

Thus, for any $h\in\mathcal{P}(X\times S^1)$, $\overline{n_-(h)} * n_+(h)$ is trivial and so

$$n_+(h) = -\overline{n_-(h)} \in \mathcal{P}(X\times S^1\times I).$$

□

Thus, in the Hatcher–Wagoner–Igusa setting, we have a duality formula for our nilpotent pseudoisotopies:

$$[n_+(h)] = (-1)^{n-1}\overline{[n_-(h)]}.$$

Proposition 6.9 $\Sigma\mathcal{N}\mathcal{P}(X) = \mathcal{N}_+\mathcal{P}(X)\oplus\mathcal{N}_-\mathcal{P}(X)$.

Proof Consider the commutative diagram below, where

$$\begin{aligned} d(n_+(h), n_-(h)) &= [n_+(h)] + (-1)^{n-1}\overline{[n_+(h)]} = [\Sigma(n(h))] \\ &= [n_-(h)] + (-1)^{n-1}\overline{[n_-(h)]} : \end{aligned}$$

$$\begin{tikzcd} & \mathcal{N}_+\mathcal{P}(X)\times\mathcal{N}_-\mathcal{P}(X) \arrow[d, "d"] \\ \mathcal{P}(X\times S^1) \arrow[ur, "(n_+,n_-)"] \arrow[r, "\Sigma n"'] & \Sigma\mathcal{N}\mathcal{P}(X) \end{tikzcd}$$

If $h\in\mathcal{P}_b(X\times S^1)$ so that $\Sigma n(h)$ is trivial, then $n(h)$ must also be trivial and thus h is relaxed. Then $n_+(h)$ and $n_-(h)$ are both isotopic to the identity so the map d is an injection.

Let $\Sigma g\in\Sigma\mathcal{N}\mathcal{P}(X)$. Then

$$g\in \mathrm{Ker}(r:\mathcal{P}(X\times S^1)\to\mathcal{R}\mathcal{P}(X\times S^1)).$$

so $r(g)$ is trivial. Therefore, g is isotopic to $n(g) = d(n_+(g), n_-(g))$. Thus, d is a bijection. Since $n(g)$ is trivial if and only if $n_+(g)$ is trivial if and only if $n_-(g)$ is trivial, $\Sigma\mathcal{N}\mathcal{P}(X)$ splits as the direct sum $\mathcal{N}_+\mathcal{P}(X)\oplus\mathcal{N}_-\mathcal{P}(X)$. □

Summarizing, if $\mathbb{P}$ denotes the group of stable pseudoisotopies.

Theorem 6.10 *For X a finite dimensional smooth manifold and dim$(X)\geq 7$, there is a splitting of groups*

$$\mathbb{P}(X\times S^1)\cong\mathbb{P}(X)\oplus\mathbb{P}_b(X\oplus\mathbb{R})\oplus\mathcal{N}_+\mathbb{P}(X)\oplus\mathcal{N}_-\mathbb{P}(X).$$

7 Pseudoisotopies and embeddings

We can identity the pseudoisotopy Nil-groups to sets of embeddings as in [6]. First, consider the set

$$\mathcal{E}mb(X \times I, X \times S^1 \times I) = \{g : X \times I \to X \times S^1 \times I : \ g \text{ is an embedding}\}/\equiv$$

where the equivalence relation is given by isotopy. More precisely, two embeddings

$$g_i : X \times I \to X \times S^1 \times I, \quad i = 0, 1,$$

are equivalent if there is an embedding

$$G : X \times I \times I \to X \times S^1 \times I$$

such that $G|_{X\times I\times\{0\}} = g_0$ and $G|_{X\times I\times\{1\}} = g_1$. We define $g \in \mathcal{E}mb(X \times I, X \times S^1 \times I)$ to be *relaxed* if

$$g(X\times I)\cap X\times\{0\}\times I = \emptyset,$$

after perhaps an isotopy.

For any such embedding, lift g to get an embedding to the infinite cyclic cover:

$$\begin{tikzcd} & X \times \mathbb{R} \times I \arrow[d] \\ X \times I \arrow[ur, "\overline{g}"] \arrow[r, "g"] & X \times S^1 \times I \end{tikzcd}$$

Let ζ denote the deck transformation on $X \times \mathbb{R} \times I$ and denote by

$$\begin{aligned} c\mathcal{E}mb(X \times I,\ X \times S^1 \times I) = \\ &\{g \in \mathcal{E}mb(X \times I, X \times S^1 \times I) : \text{ there is a diffeomorphism} \\ &\tilde{g} : X \times \mathbb{R} \times I \to X \times \mathbb{R} \times I, \tilde{g}|_{X\times\{0\}\times I} = g, \zeta \circ \tilde{g} = \tilde{g} \circ \zeta\}, \end{aligned}$$

again defined up to isotopy. Equivalently, the embeddings in this subset of $\mathcal{E}mb(X \times I, X \times S^1 \times I)$ are those which form an h-cobordism between $X \times \{0\} \times I$ and $\overline{g}(X \times I)$ in $X \times \mathbb{R} \times I$.

The transfer map in the S^1-direction induces a transfer map:

$$\mathrm{tr}^s : c\mathcal{E}mb(X \times I, X \times S^1 \times I) \to c\mathcal{E}mb(X \times I, X \times S^1 \times I).$$

It is the lift of the embedding:

$$\begin{tikzcd} & X \times S^1 \times I \arrow[d, "\times s"] \\ X \times I \arrow[ur, "\mathrm{tr}^s(g)"] \arrow[r, "g"] & X \times S^1 \times I \end{tikzcd}$$

We define a projection map

$$p : \mathcal{P}_b(X \times S^1) \to c\mathcal{E}mb(X \times I, X \times S^1 \times I), \quad \text{by } p(h) = h|_{X\times\{0\}\times I},$$

so $p(h)$ is the restriction of the pseudoisotopy h to a single fiber. Note that $p(h)$ is the identity on $X \times \{0\} \times \{0\}$ and the region $A \subseteq X \times \mathbb{R} \times I$ between $X \times \{0\} \times I$ and $\overline{p(h)}(X \times I)$ is a h-cobordism.

Proposition 7.1 *For $g, g' \in c\mathcal{E}mb(X \times I, X \times S^1 \times I)$, let*

$$g' \sim g \Longleftrightarrow g'(X \times I) \subseteq (X \times S^1 \times I) - g(X \times I).$$

Then "$\sim$" is an equivalence relation.

Proof Let $g, g' \in c\mathcal{E}mb(X \times I, X \times S^1 \times I)$. If $g'(X \times I) \subseteq (X \times S^1 \times I) - g(X \times I)$, then $g(X \times I) \subseteq (X \times S^1 \times I) - g'(X \times I)$ so the relation is symmetric. We can use a small isotopy on a collar neighborhood of $g(X \times I)$ to push $g(X \times I)$ off itself to see that the relation is reflexive.

To see that the relation is transitive, consider embeddings $g \sim f$ and $f \sim h$. Then

$$f(X \times I) = \tilde{f}(X \times \{0\} \times I) \subseteq (X \times S^1 \times I) - g(X \times I) = \tilde{g}(X \times (0, 1) \times I).$$

Thus there is an ϵ with $0 < \epsilon < 1$ so that:

$$\tilde{f}(X \times [0, \epsilon) \times I) \subseteq \tilde{g}(X \times (0, 1) \times I).$$

If $h \sim f$, then $\tilde{h}(X \times \{0\} \times I) \subseteq \tilde{f}(X \times (0, 1) \times I)$. Then there is a δ with $0 < \delta < 1$ so that $\tilde{h}(X \times \{0\} \times I) \subseteq \tilde{f}(X \times (0, \delta) \times I)$. Note that there is an isotopy of the identity on $X \times S^1 \times I$ taking $X \times (0, \delta) \times I$ to $X \times (0, \epsilon) \times I$. Composing h with this isotopy to get h', we have

$$\tilde{h}'(X \times \{0\} \times I) \subseteq \tilde{f}(X \times (0, \epsilon) \times I) \subseteq \tilde{g}(X \times (0, 1) \times I)$$

and so $h = h' \sim g$. □

Let

$$i : X \times I \to X \times S^1 \times I, \quad i(x, t) = (x, 0, t)$$

be the inclusion map.

Corollary 7.2 *Let $g \in c\mathcal{E}mb(X \times I, X \times S^1 \times I)$. Then $g \sim i$ if and only if g is relaxed.*

Proof Note that $g \sim i$, if and only if

$$g(X \times I) \subseteq X \times (0, 1) \times I \subseteq X \times S^1 \times I.$$

The result follows from the definitions. □

Let $\mathcal{CE}mb(X \times I, X \times S^1 \times I)$ denote the subset $c\mathcal{E}mb(X \times I, X \times S^1 \times I)$ modulo this equivalence relation. The transfer map on $c\mathcal{E}mb(X \times I, X \times S^1 \times I)$ induces a transfer map on $\mathcal{CE}mb(X \times I, X \times S^1 \times I)$, denoted by tr^s.

Corollary 7.3 *Let $g \in \mathcal{CE}mb(X \times I, X \times S^1 \times I)$. Then there is an $N \in \mathbb{N}$ such that $tr^\ell(g) \sim i$ for all $\ell \geq N$. Thus $tr^\ell(g)$ is relaxed, for all $\ell \geq N$.*

Proof Let $g \in \mathcal{CE}mb(X \times I, X \times S^1 \times I)$. The compactness of $X \times I$ implies that there is $N \in \mathbb{N}$, such that

$$\overline{g}(X \times I) \subseteq X \times [0, N] \times I \subseteq X \times \mathbb{R} \times I.$$

Thus, $\mathrm{tr}^\ell(g) \sim i$ for any $\ell \geq N$. □

Theorem 7.4 *For X a closed smooth manifold, we can identify $\mathcal{CE}mb(X \times I, X \times S^1 \times I)$ with $\mathcal{NP}(X) \subseteq \mathcal{P}_b(X \times S^1)$.*

Proof For $g \in \mathcal{CE}mb(X \times I, X \times S^1 \times I)$, we define $n'(g)$ using $\tilde{g}$ as for $n(h)$ for $h \in \mathcal{P}(X \times S^1)$ by Proposition 4.3. Note that $n(g)$ will not in general be a pseudoisotopy unless $g|_{X\times\{0\}}$ is the inclusion. It will, however, be a concordance, i.e., $n'(g) : X \times S^1 \times I \to X \times S^1 \times I$ will be a diffeomorphism. We thus define $n(g) = (g_0 \times 1)^{-1} n'(g) \in \mathcal{P}(X \times I)$ to get a map

$$n : \mathcal{CE}mb(X \times I, X \times S^1 \times I) \to \mathcal{NP}(X) \subseteq \mathcal{P}(X \times S^1).$$

We claim that n is well-defined. Let $f \sim g$ and consider $n(f)n(g)^{-1}$ which is the element of $\mathcal{NP}(X)$ defined by means of $\tilde{f}\tilde{g}^{-1}$. Since $f \sim g$,

$$\begin{aligned} &\tilde{f}(X \times \{0\} \times I) \subseteq \tilde{g}(X \times (0,1) \times I) \qquad \Longrightarrow \\ &\tilde{g}^{-1}(X \times \{0\} \times I) \subseteq \tilde{f}^{-1}(X \times (0,1) \times I) \Longrightarrow \\ &\tilde{f}\tilde{g}^{-1}(X \times \{0\} \times I) \subseteq X \times (0,1) \times I. \end{aligned}$$

Therefore $\tilde{f}\tilde{g}^{-1}$ is in $\mathcal{RP}(X \times S^1)$ and so the corresponding element $n(fg^{-1}) \in \mathcal{NP}(X)$ is isotopic to $1_{X\times S^1\times I}$. Thus $n(f)$ is isotopic to $n(g)$ and the map n is well-defined.

We further claim that the maps p and n^{-1} are inverses and so $\mathcal{NP}(X)$ may be identified with $\mathcal{CE}mb(X \times I, X \times S^1 \times I)$. For $g \in \mathcal{CE}mb(X \times I, X \times S^1 \times I)$,

$$p(n(g)^{-1}) = n(g)^{-1}|_{X\times\{0\}\times I}.$$

By the construction of $n(g)$, note that

$$\tilde{g}(X \times \{0\} \times I) \subseteq n(g)^{-1}(X \times (0,1) \times I).$$

Thus $p(n(g)^{-1}) \sim g$ in $\mathcal{CE}mb(X \times I, X \times S^1 \times I)$. Next, consider $h \in \widetilde{Nil}(X) \subseteq \mathcal{P}(X \times S^1)$. Since $r(h) = h \circ n(h)$ must be trivial, h is isotopic to $n(h)^{-1}$. Note that $n(h)^{-1}$ is defined by means of $\tilde{h}$ while $n^{-1}(p(h))$ is defined by $\tilde{h}|_{X\times\{0\}\times I}$. These agree on each $X \times \{i\} \times I$, so $\tilde{h}$ and $\tilde{h}|_{X\times\{0\}\times I}$ give the same equivalence class in $\mathcal{CE}mb(X \times I, X \times S^1 \times I)$. Thus $n^{-1}(p(h))$ is isotopic to h. Therefore, p and n are inverses, so $\mathcal{CE}mb(X \times I, X \times S^1 \times I) \equiv \mathcal{NP}(X)$. □

Remark 7.5 There is not an obvious geometric construction of a group operation on $\mathcal{CE}mb$ $(X \times I, X \times S^1 \times I)$. That is why the above correspondence is just a bijection.

8 Pseudoisotopies in Top

Let X be a closed topological manifold. We recall a basic definition from [25]. Let B any space and

$$p_i : E_i \to B, \quad i = 0, 1,$$

be two maps. A *controlled map* from p_0 to p_1 is a map

$$g : E_0 \times [0,1) \to E_1 \times [0,1)$$

such that

(1) g is fiber preserving over $[0, 1)$.
(2) The map $\bar{g} : E_0 \times [0,1] \to E_1 \times [0,1]$ defined by

$$\bar{g}(e,t) = \begin{cases} p_1(\mathrm{pr})(g(e,t)), & 0 \le t < 1, \\ p_0(e), & t = 1. \end{cases}$$

is continuous where pr is the projection of $E_1 \times [0, 1)$ onto E_1.

We write $g^c : E_0 \to E_1$. Now we apply this definition to pseudoisotopies. Let X be a finite dimensional manifold equipped with a map $p : X \to B$. A *controlled pseudoisotopy* over B is a controlled map from the identity to the projection map to B

$$h^c : X \times I \to X \times I$$

such that the restriction $h|X \times \{0\} \times [0, 1)$ is the identity.

There is an alternate definition in [20]. The two definitions are equivalent when the base space B is a compact metric space [25]. This will be the definition that we will use in the rest of this paper. For $\varepsilon > 0$ and a control map $p : X \to B$, a ε*-pseudoisotopy* is a diffeomorphism

$$h : X \times I \to X \times I$$

such that

(1) $h|X \times \{0\}$ is the identity.
(2) $d(p\pi h, p\pi) < \varepsilon$, where π is the projection to X.

A *controlled pseudoisotopy* is a diffeomorphism

$$h : X \times I \times [0, \infty) \to X \times I \times [0, \infty)$$

such that

(1) $h|X \times \{0\} \times [0, \infty)$ is the identity.
(2) There is an decreasing sequence $\{\varepsilon_i\}_{i=0}^{\infty}$ of positive real numbers converging to 0 (*the controlling sequence* of h) such that for each $i \geq 0$,

$$h| : X \times I \times \{u\} \to X \times I \times \{u\}$$

is a ε_i-pseudoisotopy whenever $u \geq i$.

We write $\mathcal{P}_c^{\mathrm{top}}(X \to B)$ for the subgroup of controlled pseudoisotopies.

The following sequence is split exact:

$$0 \to \mathcal{P}^{\mathrm{top}}(X \times I) \xrightarrow{i} \mathcal{P}_c^{\mathrm{top}}(X \times S^1 \to S^1) \xrightarrow{\mathrm{tr}^\infty} \mathcal{P}_b^{\mathrm{top}}(X \times \mathbb{R}^1) \to 0.$$

This result is from [25], Sect. 1.5 (also [4,6]) for finite dimensional manifolds.

Using Lemma 3.1, we compare the relaxed and the controlled pseudoisotopy groups.

Proposition 8.1 *There is an isomorphism*

$$\rho : \mathcal{P}_c^{\mathrm{top}}(X \times S^1 \to S^1) \to \mathcal{RP}^{\mathrm{top}}(X \times S^1).$$

Proof Let $h_t : X \times S^1 \times I \to X \times S^1 \times I$, $0 \leq t < \infty$, be a controlled pseudoisotopy over S^1. Let π_2 denote the projection to S^1. Let $\{\varepsilon_i\}_{i=0}^{\infty}$ be the controlling sequence. Since $\{\varepsilon_i\}_{i=0}^{\infty}$ converges to 0, there is an $\varepsilon_j < \frac{1}{4}$. Then h_u satisfies:

$$d(\pi_2 h_u(x, s, t), s) < \varepsilon_j < \frac{1}{4}, \quad \text{for all } \ u > j.$$

Fix $u > j$. The claim is that h_u is relaxed. To see this, notice that

$$h_u(X \times \{1\} \times I) \subset X \times (-\varepsilon_j, \varepsilon_j) \times I.$$

Define an isotopy

$$g_r : X \times S^1 \times I \to X \times S^1 \times I, \quad \text{by } g_r(x, s, t) = (x, s + 2\varepsilon_j r, t).$$

Then $g_0 \circ h_u = h_u$ and

$$g_1(h_u(X \times \{0\} \times I)) \subset X \times (\varepsilon_j, 3\varepsilon_j) \times I.$$

Thus $g_1(h_u(X \times \{0\} \times I) \cap (X \times \{0\} \times I) = \emptyset$ and h_u is relaxed. Define $\rho(h_t) = h_u$. It is immediate from the construction that any two choices of u give isotopic images. Also, ρ does not depend on the isotopy class of h_t.

Also, it is clear that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{P}^{\text{top}}(X \times I) \arrow[r, "i"] \arrow[d, "\text{id}"] & \mathcal{P}_c^{\text{top}}(X \times S^1 \to S^1) \arrow[r, "\text{tr}^\infty"] \arrow[d, "\rho"] & \mathcal{P}_b^{\text{top}}(X \times \mathbb{R}) \arrow[r] \arrow[d, "\text{id}"] & 0 \\
0 \arrow[r] & \mathcal{P}^{\text{top}}(X \times I) \arrow[r, "i"] & \mathcal{RP}^{\text{top}}(X \times S^1) \arrow[r, "\text{tr}^\infty"] & \mathcal{P}_b^{\text{top}}(X \times \mathbb{R}) \arrow[r] & 0
\end{tikzcd}$$

Then the result follows from the five lemma. □

Remark 8.2

(1) The Squeezing Lemma or Sucking Principle implies that there is a number $\varepsilon > 0$ so that if pseudoisotopies (or fibrations, or homotopy equivalences) are ε-controlled and the control map has good local properties (being a bundle or just an approximate fibration) then the pseudoisotopy is controlled. There is a theme that appears in problems ‘over the circle’. In this case, the number ε seems to be equal to 2π, in other words the length of the circumference. In other words, if the map to S^1 does nor wrap around the circle completely, then the pseudoisotopy is controlled over S^1. The proof should follow by pushing the pseudoisotopy to transverse shorter and shorter arcs. With that in mind, a relaxed pseudoisotopy is controlled because it does not wrap around S^1. We could make these remarks precise and that is why the proof of Proposition 8.1 uses the splittings of the controlled and the relaxed pseudoisotopy groups.

(2) If we combine the result of Proposition 5.1, Part (2), with Proposition 8.1 we get the Sucking or Squeezing Principle for the pseudoisotopy groups. More precisely, it states that given $h \in \mathcal{P}^{top}(X \times S^1)$, $\text{tr}^\ell(h)$ is ‘more controlled’ in the S^1-direction than h. For large enough ℓ, the Sucking Principle implies that $\text{tr}^\ell(h)$ is controlled, i.e., $\text{tr}^\ell(h) \in \mathcal{P}_c^{top}(X \times S^1 \to S^1)$.

Using Theorem 5.4 and Proposition 8.1 we get the following.

Theorem 8.3 *Let X be a closed finite dimensional topological manifold. Then*

$$\mathcal{P}^{top}(X \times S^1) \cong \mathcal{RP}^{\text{top}}(X \times S^1) \oplus \mathcal{NP}(X) \cong \mathcal{P}_c^{\text{top}}(X \times S^1 \to S^1) \oplus \mathcal{NP}(X)$$

given by $h = \phi(r(h)) + n(h)$. Furthermore

$$\mathcal{P}^{\text{top}}(X \times S^1) \cong \mathcal{P}^{\text{top}}(X \times I) \oplus \mathcal{P}_b^{\text{top}}(X \times \mathbb{R}) \oplus \mathcal{NP}(X).$$

The constructions in Sect. 6 work for both the smooth and the topological category.

Theorem 8.4 *For X a finite dimensional topological manifold and dim(X) greater than or equal to the stable range for topological manifolds [5], there is a splitting of groups*

$$\mathbb{P}^{\text{top}}(X \times S^1) \cong \mathbb{P}^{\text{top}}(X) \times \mathbb{P}_b^{\text{top}}(X \times \mathbb{R}) \times \mathcal{N}_+\mathbb{P}^{\text{top}}(X) \times \mathcal{N}_-\mathbb{P}^{\text{top}}(X).$$

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