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A Split Exact Sequence of Equivariant K-Groups of Virtually Nilpotent Groups

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Abstract. Let Γ be an extension of a torsion free nilpotent group by a finite group G, of odd order. Then Γ admits a cocompact proper action on $\mathbb{R}^n$. This action determines an action of G on a nil-manifold by isometries. In this paper, the equivariant K-theory of the finite group action is studied and it is shown that the 'forget control' map is a split monomorphism.

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1. Introduction

Let Γ be a finite extension of a finitely presented, torsion free nilpotent group N. Then Γ admits a cocompact, properly discontinuous action on the euclidean space of dimension equal to the virtual cohomological dimension of Γ. Since N acts freely on the euclidean space, the above action induces an action of the finite group $G = \Gamma/N$ on a nil-manifold M_Γ. Steinberger–West, [22], [23], introduced topological equivariant K-theory for studying equivariant h-cobordisms in this situation. For calculating those groups, they proved that the equivariant topological K-groups fit in an exact sequence involving the equivariant PL K-theory groups and their controlled analogues

$$\tilde{K}^{\mathrm{PL}}_{i,G}(M_\Gamma)_c \to \tilde{K}^{\mathrm{PL}}_{i,G}(M_\Gamma) \to \tilde{K}^{\mathrm{Top}}_{i,G}(M_\Gamma) \to \tilde{K}^{\mathrm{PL}}_{i-1,G}(M_\Gamma)_c \to \tilde{K}^{\mathrm{PL}}_{i-1,G}(M_\Gamma).$$

In this paper we study the 'forget control' between the equivariant PL K-groups. The main result of the paper is that, under certain assumptions, the 'forget control' map is a split monomorphism.

MAIN THEOREM. *Let Γ be an extension of a finitely presented, torsion free, nilpotent group by a finite group G of odd order and M_Γ is the G-manifold defined above. Then the 'forget control' map*

$$\tilde{K}^{\mathrm{PL}}_{i,G}(M_\Gamma)_c \to \tilde{K}^{\mathrm{PL}}_{i,G}(M_\Gamma)$$

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is a split monomorphism for all $i \leqslant 1$. In particular, the above exact sequence is reduced to a collection of short exact sequences.

Notice that the map in the main theorem is rarely an isomorphism because there are summands of the PL K-group that are isomorphic to K-theory Nil-groups of finite groups and so they are not controlled.

Connolly–Koźniewski, [4], have shown that the forget control map is a monomorphism for Γ a crystallographic group. The equivariant PL K-groups split in a direct sum of the K-groups of the strata. In the Main Theorem, if we consider the summand corresponding to the top stratum, we see that the Main Theorem generalizes the theorem in the announcement in [20].

The Main Theorem can be considered as an analogue of the Equivariant Novikov Conjecture in K-theory ([15]). Also, the result in the Main Theorem is connected with the equivariant K-theory rigidity conjecture and the Equivariant Borel conjecture. We now formulate the equivariant K-theory rigidity conjecture, a precise statement of the equivariant Borel conjecture is in [4]. Let Γ be a cocompact discrete subgroup of the group of isometries of a Hadamard manifold $\tilde{M}$. Let Γ_0 be a torsion free subgroup of finite index in Γ and $G = \Gamma/\Gamma_0$. Then $M_\Gamma = \tilde{M}/\Gamma_0$ is a compact G-manifold.

EQUIVARIANT K-THEORY RIGIDITY CONJECTURE. *With the above notation, assume that*

(1) Γ *does not contain subgroups isomorphic to $D_\infty \times \mathbb{Z}$ or to $H \times_\alpha K$ where K is the fundamental group of the Klein bottle and H is a finite group whose order is divisible of the second power of some prime*
(2) *the action of G on M_Γ does not have fixed point sets of dimension two, three, or four and that proper inclusions of fixed point set components have dimension bigger than two. Then*

$$\hat{H}^*(\mathbb{Z}/2\mathbb{Z}; \tilde{K}_{i,G}^{\mathrm{Top},\rho}(M_\Gamma)) = 0, \quad \text{for all} \quad i \leqslant 1,$$

where $\mathbb{Z}/2\mathbb{Z}$ acts on $\tilde{K}_{i,G}^{\mathrm{Top},\rho}(M_\Gamma)$ by inverting the G-h-cobordisms.

In [7], we prove the Equivariant K-theory Rigidity Conjecture for $i \leqslant 0$ for a large class of cocompact subgroups of groups of isometries of Hadamard manifolds without 2-torsion elements. For the connection between the Equivariant K-theory Conjecture and the Equivariant Borel Conjecture, see [4].

Condition (1) in the statement of the equivariant rigidity K-theory conjecture is necessary, as it has been shown in [5]. Condition (2) is required for formulating a geometrically reasonable problem.

The proof of the Main Theorem combines the ideas developed in [4] with the ideas in [10–12]. We compare the controlled groups with the K-theory groups with control to a crystallographic manifold. The above control map is defined using the machinery of *fibering apparatus* defined in [10], [11] (see also [26]), [20]). Notice that the idea for comparing the two control maps has been used before in [20] for proving a similar result. The advantage in using crystallographic manifolds is that they admit expansive maps which are used to make the control tighter.

Using the exact sequence developed in [22], [21], we can reduce the problem in the Main Theorem to the study of PL equivariant Whitehead groups and their controlled analogues. Combining this idea with the ideas developed in the last paragraph, we show that it is enough to consider controlled groups where the control space is a crystallographic manifold. The rest of the proof applies the ideas developed in [4].

The above discussion suggests that the equivariant topological K-theory groups of finite group actions on non-positively curved manifolds are essentially built by Nil-groups. Therefore the study of the properties of Nil-groups is very important in understanding equivariant rigidity phenomena in this setting. This idea is applied in [7] where calculations on Nil-groups ([3], [6]) are applied for calculating the exponents of the lower equivariant topological K-groups.

2. Equivariant K-Theory

Let G be a finite group and X a finite G-CW complex. Then $\mathrm{Wh}_G^{\mathrm{Top}}(X)$, the topological equivariant Whitehead group of X, is the group of equivalence classes of G-strong deformation retract pairs (Y, X), where Y is a compact G-ANR. The equivalence relation is generated by G-CE maps, rel$X$, and the group operation is given by unions over X, [21], [22], [23]. The PL analogue of the above group, $\mathrm{Wh}_G^{\mathrm{PL}}(X)$ is constructed similarly. In this case we consider pairs (Y, X), where (Y, X) is a relative finite G-CW complex, such that there is a cellular G-strong deformation retraction from Y to X, and the equivalence relation is generated by G-CEPL maps. Both the PL and Top equivariant Whitehead groups admit a direct sum decomposition with one summand for each conjugacy class of subgroups of G, [21], [23].

Generalizing the methods developed by Chapman, Steinberger and West defined the controlled analogue of the equivariant PL group. Let B be a finite dimensional G-metric space and $p : X \to B$ be a G-map. If $\varepsilon > 0$, a $p^{-1}(\varepsilon)$-G-strong deformation retraction means a G-deformation retraction whose tracks have diameter less than ε when they are measured in B. The set of elements of $\mathrm{Wh}_G^{\mathrm{PL}}(X)$ that are $p^{-1}(\varepsilon)$-G-strong deformation retractions and have inverses that are also $p^{-1}(\varepsilon)$-G-strong deformation retractions form a group. Notice that if $\varepsilon' > \varepsilon$ then every $p^{-1}(\varepsilon)$-G-strong deformation retraction is a $p^{-1}(\varepsilon')$-G-strong deformation retraction. Thus the above groups form a directed system. The controlled Whitehead group, $\mathrm{Wh}_G^{\mathrm{PL}}(X)_p$ is the inverse limit of this system. When p is an equivariant simplicial p-NDR then the above inverse system is stable ([18], [21–23]). The controlled Whitehead groups admit a direct sum decomposition similar to the one in the uncontrolled case. We write $\mathrm{Wh}_G^{\mathrm{PL}}(X)c$ whenever the control map is the identity.

The lower K_i-groups, $i < 1$, are defined as the groups of transfer invariant elements of the Whitehead groups of $X \times T^{1-i}$, where the transfer is over the finite covers of T^{1-i} (the G action on T^{1-i} is trivial).

The main calculational tool in equivariant K-theory is the five term exact sequence given in [21], [23], [4]

$$\tilde{K}^{\mathrm{PL}}_{i,G}(X)_c \to \tilde{K}^{\mathrm{PL}}_{i,G}(X) \to \tilde{K}^{\mathrm{Top}}_{i,G}(X) \to \tilde{K}^{\mathrm{PL}}_{i-1,G}(X)_c \to \tilde{K}^{\mathrm{PL}}_{i-1,G}(X)$$

for $i \leqslant 1$.

We define the restricted equivariant Whitehead groups $\tilde{K}^{\mathrm{Top},\rho}_{i,G}(X)$ to be the subgroups of the above groups generated by pairs (Y, X) such that $Y^H_\alpha - Y^{>H}_\alpha = \emptyset$ whenever $X^H_\alpha - X^{>H}_\alpha = \emptyset$ for all the components of fixed point sets. The restricted equivariant Whitehead groups admit a direct sum decomposition with one summand for each conjugacy class of the isotropy subgroups of G. There is a five term exact sequence for the restricted K-groups, ([21], [4]). The restricted equivariant Whitehead groups are isomorphic to the isovariant Whitehead groups and they classify isovariant topological G-h-cobordisms over X when X is a G-manifold ([21]).

In (X, X') is a pair we define the relative Whitehead groups to be the subgroups of the Whitehead groups of X consisting of those G-strong deformation retractions that are the identity on X' ([4]).

Let G act by isometries on a connected compact manifold M. Let $\Gamma = \pi_1(EG \times_G M)$. Then there is an exact sequence $1 \to \Gamma_0 \to \Gamma \to G \to 1$, where $\Gamma_0 = \pi_1(M)$. Notice also that Γ can be identified with $\pi_1(M - \sigma M)$ where σM is the singular set of the G action on M. Following [4], we define $\mathrm{Wh}^{\mathrm{Top}}(\Gamma) = \mathrm{Wh}^{\mathrm{Top},\rho}_G(M, \sigma M)$. If $p : M \to M'$ is a G-map then we define $\mathrm{Wh}(\Gamma)_p = \mathrm{Wh}^{\mathrm{PL},\rho}_G(M, \sigma M)_p$. The lower controlled and topological K-groups of Γ, $K_i(\Gamma), i < 1$, are defined as the subgroups of the corresponding Whitehead group of $M \times T^{1-i}$, where T^{1-i} is a trivial G-torus, consisting of the transfer invariant elements as before. In [4] it was shown that there is an exact sequence, for $i \geqslant -1$

$$\tilde{K}_{-i}(\Gamma)_c \to \tilde{K}_{-i}(\Gamma) \to \tilde{K}^{\mathrm{Top}}_{-i}(\Gamma) \to \tilde{K}_{-i-1}(\Gamma)_c \to \tilde{K}_{-i-1}(\Gamma),$$

where $\tilde{K}_1 = \mathrm{Wh}$, $\tilde{K}_{-i} = K_{-i}$ for $i > 0$. The above exact sequence is the restriction of the Steinberger–West exact sequence to the summands corresponding to the top stratum ([4]). If the control map p is chosen to be the natural projection $p : EG \times_G M \to M/G$ then $\mathrm{Wh}(\Gamma)_c$ is the group $H_1(M/G; \mathcal{W}h(p))$ defined in [17] and [18] ([4]).

3. Virtually Nilpotent Groups and Fibering Apparatus

Let Γ be a virtually nilpotent group i.e. there is an exact sequence

$$1 \to N \to \Gamma \to G \to 1, \qquad (*)$$

where N is a finitely presented torsion free nilpotent group and G is finite. Since N has finite cohomological dimension, Γ has finite virtual cohomological dimension. Let $n = \mathrm{vcd}(\Gamma)$.

LEMMA 3.1. Γ *acts cocompactly and properly discontinuously on* $\mathbb{R}^n$.

The proof is in [10] for the torsion free case and in [26] for the general case. Notice that in the case of a finite group Γ ($\mathrm{vcd}(\Gamma) = 0$), $n = 0$ and Γ acts on a point.

DEFINITION 3.1.1. *Let* Γ *be as above. A fibering apparatus for* $(\Gamma, \mathbb{R}^n)$ *is a triple* $\mathcal{A} = (\hat{\Gamma}, \phi, f)$ *where*

(1) *A crystallographic group* $\hat{\Gamma} \subset E(m)$, $m > 0$, *when vcd*$(\Gamma) > 0$, *and* $\hat{\Gamma}$ *fits into an exact sequence*
$$1 \to A \to \hat{\Gamma} \to F \to 1, \tag{**}$$
where $A \simeq \mathbb{Z}^m$ *is the translation subgroup,* F *is finite, and the action of* F *on* A*, determined by the exact sequence, is effective.* $\hat{\Gamma}$ *is the trivial group if* Γ *is finite.*
(2) *A group epimorphism* $\phi : \Gamma \to \hat{\Gamma}$.
(3) *A* ϕ*-equivariant map* $f : \mathbb{R}^n \to \mathbb{R}^m$ *which is a fiber bundle with fiber* $\mathbb{R}^{n-m}$.
(4) *The map* ϕ *induces a map of short exact sequences*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & N & \longrightarrow & \Gamma & \longrightarrow & G & \longrightarrow & 1 \\
 & & \big\downarrow{\scriptstyle \phi_0} & & \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle \phi'} & & \\
1 & \longrightarrow & A & \longrightarrow & \hat{\Gamma} & \longrightarrow & F & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
 & & 1 & & 1 & & 1. & &
\end{array}$$

PROPOSITION 3.2. *If* Γ *is as in (*), then there is a fibering apparatus for* $(\Gamma, \mathbb{R}^n)$.

Proof. The proof of the Proposition is in [10] for the torsion free case and in [26] in the general case.

Remarks 3.2.1. (i) If $M_\Gamma = \mathbb{R}^n/N$, $M_{\hat{\Gamma}} = \mathbb{R}^m/A$ then ϕ induces a ϕ'-equivariant map $f' \colon M_\Gamma \to M_{\hat{\Gamma}}$.

(ii) Let $v\in\mathbb{R}^n$ and $\hat{\Gamma}_v$ be the isotropy group of the $\hat{\Gamma}$ action on $\mathbb{R}^m$. Let $\Gamma_v = \phi^{-1}(\hat{\Gamma}_v)$. Notice that $\hat{\Gamma}_v$ is finite, Γ_v is virtually nilpotent, and $\mathrm{vcd}(\Gamma_v) = n - m$. Let $1 \to N_v \to \Gamma_v \to G_v \to 1$ be the exact sequence

$$1 \to N\cap\phi^{-1}(\hat{\Gamma}_v) \to \phi^{-1}(\hat{\Gamma}_v) \to {\phi'}^{-1}(\hat{\Gamma}_v) \to 1.$$

Also notice that there is a canonical cocompact action of Γ_v on $f^{-1}(v) \cong \mathbb{R}^{n-m}$.

We will try to compare the control K-theory groups of the manifold M_Γ with control map the identity and the K-theory groups with control map $f' : M_\Gamma \to M_{\hat{\Gamma}}$.

PROPOSITION 3.3. *There is a spectral sequence*

$$E^2_{i,j} = H^{lf}_i(M_{\hat{\Gamma}}/F;\, \tilde{K}_j(\Gamma_v)_c), \quad \textit{for } j \leqslant 1,$$

where $E^\infty_{i,j}$ abuts to $\tilde{K}_{i+j}(\Gamma)_c$ for $i + j \leqslant 1$.

Proof. The spectral sequence is derived from 'the change of control' spectral sequence discussed in [17], 2.6 and [26], Proposition 2.3. If $s : EG\times_G M_\Gamma \to M_\Gamma/G$ is the orbit map, then $\tilde{K}_{i+j}(\Gamma)_c = H^{lf}_{i+j}(M_\Gamma/G;\, \mathcal{W}h(s))$ where $\mathcal{W}h$ is the spectrum introduced in [17] (or in [18]).

Consider the sequence of the two stratified systems of fibrations

$$EG\times_G M_\Gamma \xrightarrow{s} M_\Gamma/G \xrightarrow{\hat{f}} M_{\hat{\Gamma}}/F.$$

Using the calculations in [17], [26] we obtain

$$E^2_{i,j} = H_i\left(M_{\hat{\Gamma}}/F;\, \bigcup_{x\in M_{\hat{\Gamma}}/F} H^{lf}_j(\hat{f}^{-1}(x);\, \mathcal{W}h(s|_{\hat{f}^{-1}(x)}))\right) \Rightarrow \tilde{K}_{i+j}(\Gamma)_c,$$

which can be written, using the calculations in [18], Chapter. 8

$$\begin{aligned} E^2_{i,j} &= H_i\left(M_{\hat{\Gamma}}/F;\, \bigcup_{x\in M_{\hat{\Gamma}}/F} \tilde{K}_j(\Gamma_v)_c\right) \\ &= H_i(M_{\hat{\Gamma}}/F;\, \tilde{K}_j(\Gamma_v)_c), \end{aligned}$$

where $v\in\mathbb{R}^m$ and $x = [v]$ under the identification $\mathbb{R}^m/\hat{\Gamma} = M_{\hat{\Gamma}}/F$.

Remarks 3.3.1. (i) The homology groups appearing in the E^2-term of the spectral sequence can be identified with Bredon homology as in [4]. Consider the coefficient system, in the sense of Bredon ([1]) which assigns to each subgroup H of $\hat{\Gamma}$ the group $\tilde{K}_i(\phi^{-1}(H))_c$ and to each inclusion $K \subset H$ the induction map

$\mathrm{ind}_K^H : \tilde{K}_i(\phi^{-1}(K))_c \to \tilde{K}_i(\phi^{-1}(H))_c$ ([4]). The above coefficient system defines an equivariant homology theory on $M_{\hat{\Gamma}}$ ([1]). Then

$$H_i^{lf}(M_{\hat{\Gamma}}/F;\ \tilde{K}_j(\Gamma_v)_c) \cong H_i^F(M_{\hat{\Gamma}};\ \tilde{K}_j(\Gamma_v)_c).$$

(ii) Proposition 2.6 in [17] and Proposition 2.3 in [26] provide a homotopy equivalence of spectra whose homotopy groups are the homology groups used in the proof of Proposition 3.3. In particular, it was proved that there is a homotopy equivalence of spectra

$$f_* : \mathbb{H}_*(M_\Gamma/G; \mathcal{W}h(s)) \to \mathbb{H}_*(M_{\hat{\Gamma}}/F; \mathbb{H}_*(\hat{f}^{-1}(x); \mathcal{W}h(s|_{\hat{f}^{-1}(x)})).$$

Let $\hat{\Gamma}$ be a crystallographic group as in $(**)$ and $M_{\hat{\Gamma}}/F$ be the crystallographic torus. An *s-expansive map*, $g : M_{\hat{\Gamma}} \to M_{\hat{\Gamma}}$, is an F-map such that:

(1) $g_* : H_1(M_{\hat{\Gamma}};\ \mathbb{Z}) \simeq A \to H_1(M_{\hat{\Gamma}};\ \mathbb{Z}) \simeq A$ is multiplication by s.

(2) The map $\tilde{g} : \widetilde{M_{\hat{\Gamma}}} \to \widetilde{M_{\hat{\Gamma}}}$ induced on the universal covers, is a diffeomorphism which expands distances by a factor of s.

The map $g : M_{\hat{\Gamma}} \to M_{\hat{\Gamma}}$ induces a map $\mathrm{id}{\times}g : EF{\times}_F M_{\hat{\Gamma}} \to EF{\times}_F M_{\hat{\Gamma}}$.

We write $\alpha = (\mathrm{id}{\times}g)_* : \hat{\Gamma} \to \hat{\Gamma}$. Notice that α is defined up to conjugation by an element of A, and it is an s-expansive map in the classical sense of Epstein–Schub, [8]. Also the map $\tilde{g}$ is α-equivariant. The existence of s-expansive maps was shown in [8], [4].

LEMMA 3.4. *For each $s \equiv 1 \bmod |F|$, s-expansive maps exist.*

Let Γ be as in (*) and $\mathcal{A} = (\hat{\Gamma}, \phi, f)$ be a fibering apparatus for $(\Gamma, \mathbb{R}^n)$ and let $g : M_{\hat{\Gamma}} \to M_{\hat{\Gamma}}$ be an s-expansive map. Form the pull-back

$$\begin{tikzcd} M_g \arrow[r, "f_g"] \arrow[d, "g_\Gamma"] & M_{\hat{\Gamma}} \arrow[d, "g"] \\ M_\Gamma \arrow[r, "f'"] & M_{\hat{\Gamma}}. \end{tikzcd} \qquad (\#)$$

Then M_g is a compact G-manifold, $\pi_1(M_g) = N_g$ is nilpotent, because this is a subgroup of N, g_Γ is a G-map, and f_g is an ϕ'-map, where $\phi' : G \to F$. Therefore g induces a transfer map $g_c^! : \tilde{K}_i(\Gamma)_c \to \tilde{K}_i(\Gamma_g)_c$ where $\Gamma_g = \pi_1(EG{\times}_G M_g)$ is a virtually nilpotent group.

LEMMA 3.5. *The triple $\mathcal{A}' = (\hat{\Gamma}, \alpha\phi|_{\Gamma_g}, \tilde{g}f)$ is a fibering apparatus for the pair $(\Gamma_g, \mathbb{R}^n)$, where $\tilde{g}$ is the lift of g on the universal covers.*

Proof. Notice that the diagram (#) is a pull-back diagram where the horizontal maps are finite coverings. Therefore $N/N_g \simeq A/\alpha(A) \simeq \mathbb{Z}^m/s\mathbb{Z}^m$ and the isomorphism is induced by ϕ. So $\phi(N_g) = \alpha(A)$. Also, by definition, $\Gamma/N \simeq \Gamma_g/N_g \simeq G$, and $\hat{\Gamma}/A \simeq \alpha(\hat{\Gamma})/\alpha(A) \simeq F$. Then the epimorphism $\phi' : G \to F$ induces an epimorphism $\Phi : \Gamma_g/N_g \to \alpha(\hat{\Gamma})/\alpha(A)$ defined by $\Phi(\gamma N_g) = \phi(\gamma)\alpha(A)$. Therefore $\phi(\Gamma_g) = \alpha(\hat{\Gamma})$. So the map $\Gamma_g \xrightarrow{\phi|} \hat{\Gamma} \xrightarrow{\alpha} \hat{\Gamma}$ is an epimorphism. Since the map $\tilde{g}$ is α-equivariant, it follows that the map $\tilde{g}f$ is a $\alpha\phi|_{\Gamma_g}$-equivariant map which is also a fiber bundle with fiber a Euclidean space. Condition (iii) in the definition of a fibering apparatus follows by construction. Therefore $\mathcal{A}'$ is a fibering apparatus for $(\Gamma_g, \mathbb{R}^n)$.

For $v \in \mathbb{R}^m$, with isotropy group $\hat{\Gamma}_v$, let $\Gamma'_v = (\alpha\phi)^{-1}(\hat{\Gamma}_v)$. Then as in Proposition 3.3, there is a spectral sequence

$$E^2_{i,j} = H^{lf}_i(M_{\hat{\Gamma}}/F, \tilde{K}_j(\Gamma'_v)_c), \quad \text{for } j \leqslant 1,$$

where $E^\infty_{i,j}$ abuts to $\tilde{K}_{i+j}(\Gamma_g)_c$ for $i + j \leqslant 1$.

Following the ideas in the proof of Proposition 2.13 in [4], we can show the following

PROPOSITION 3.6. *There are infinitely many $s \equiv 1 \mathrm{mod}|F|$ such that the transfer map $g^!_c : \tilde{K}_i(\Gamma)_c \to \tilde{K}_i(\Gamma_g)_c$ induced by an s-expansive map g, is a monomorphism for all $i \leqslant 1$.*

Let Γ be a virtually nilpotent group as in $(*)$, and $\mathcal{A} = (\hat{\Gamma}, \phi, f)$ be a fibering apparatus for the natural action of Γ on $\mathbb{R}^n$. Let $\tilde{K}_i(\Gamma)_f$, for $i \leqslant 1$, be the control K-group of Γ with control map $f' : M_\Gamma \to M_{\hat{\Gamma}}$. As before, $\tilde{K}_i(\Gamma)_f$ is the summand of $\tilde{K}_i(M_\Gamma)_{f'}$ corresponding to the trivial group (for the decomposition of the equivariant control K-groups see [21], p. 77). As before $\tilde{K}_i(\Gamma)_f$ is the group $H^{lf}_i(M_{\hat{\Gamma}}/F, \mathcal{W}h(\hat{f}s))$ where

$$EG \times_G M_\Gamma \xrightarrow{s} M_\Gamma/G \xrightarrow{\hat{f}} M_{\hat{\Gamma}}/F.$$

PROPOSITION 3.7. *With the above notation:*

(a) *There is a spectral sequence*

$$E^2_{i,j} = H_i(M_{\hat{\Gamma}}/F, \tilde{K}_j(\Gamma_v)), \quad \textit{for } j \leqslant 1$$

where $E^\infty_{i,j}$ abuts to $\tilde{K}_{i+j}(\Gamma)_f$ for $i + j \leqslant 1$.

(b) *There are infinitely many $s \equiv 1 mod|F|$ such that the transfer*

$$g^!_f : \tilde{K}_i(\Gamma)_f \to \tilde{K}_i(\Gamma_g)_{gf'}$$

induced by an s-expansive map g, is a monomorphism for $i \leqslant 1$.

Proof. Part (a) is a special case of the spectral sequence in [18], Chapter. 8. The proof of part (b) is the same as the proof of Proposition 2.13 in [4].

The map f induces a homomorphism $f_\# : \tilde{K}_i(\Gamma)_c \to \tilde{K}_i(\Gamma)_f$. The map $f_\#$ is really the 'forget control' map in the directions of the fibers of f.

LEMMA 3.8. *The forget control map induces a map*

$$H_i(M_{\hat{\Gamma}}/F; \tilde{K}_j(\Gamma_v)_c) \to H_i(M_{\hat{\Gamma}}/F; \tilde{K}_j(\Gamma_v)),$$

which in turn induces the map $\bar{f}_\# : \tilde{K}_j(\Gamma_v)_c \to \tilde{K}_j(\Gamma)_f$ *on the limit of the spectral sequences, where* $\bar{f}_\#$ *is the map induced by* $f_\#$ *on the associated graded groups.*

Proof. For each $v \in \mathbb{R}^m$, consider the assembly map of spectra, where $x = [v] \in M_{\hat{\Gamma}}$

$$A_v : \mathbb{H}_*(\hat{f}^{-1}(x); \mathcal{W}h(s|_{\hat{f}^{-1}(x)})) \to \mathcal{W}h(s^{-1}\hat{f}^{-1}(x)).$$

The map A_v induces the forget control map to the homotopy groups of the spectra. The maps A_v induce a map between spectra

$$A : \mathbb{H}_*(M_{\hat{\Gamma}}/F; \mathbb{H}_*(\hat{f}^{-1}(x); \mathcal{W}h(s|_{\hat{f}^{-1}(x)}))) \to \mathbb{H}_*(M_{\hat{\Gamma}}/F; \mathcal{W}h(\hat{f}^{-1}s)).$$

Then the composite of the map f_* of Remark 3.3.1(ii) and A

$$Af_* : \mathbb{H}_*(M_\Gamma/G; \mathcal{W}h(s)) \to \mathbb{H}_*(M_{\hat{\Gamma}}/F; \mathcal{W}h(\hat{f}^{-1}s))$$

is the map $\mathbb{H}_*(\hat{f})$ induced by $\hat{f}$. This can be proved by direct calculation, using the description of the map f_* given in Proposition 2.3 in [26]. The map $f_\#$ is induced by $\mathbb{H}_*(\hat{f})$ on the homotopy groups. Therefore the map

$$H_{i,j}(\hat{f}) : H_i(M_\Gamma/G; \tilde{K}_j(s^{-1}(v))) \to H_i(M_{\hat{\Gamma}}/F; \tilde{K}_j(s^{-1}\hat{f}^{-1}(w)))$$

induced by $\hat{f}$, induces $\bar{f}_\#$ on the limit of the spectral sequences. Since $\mathbb{H}_*(\hat{f})$ can be decomposed as Af_*, the map $H_{i,j}(\hat{f})$ can be decomposed as a composition $A'H_*(f)$ of maps induced by A and f_*. But the map

$$A'_{i,j} : H_i(M_{\hat{\Gamma}}/F; \tilde{K}_j(\Gamma_v)_c) \to H_i(M_{\hat{\Gamma}}/F; \tilde{K}_j(\Gamma_v))$$

is induced by A which is induced by the forget control map on the K-groups. Therefore the forget control map on the K-groups induces a map on the E^2-terms of the spectral sequences that induces the map $\bar{f}_\#$ on the limit.

Remark 3.8.1. Using the results of [2], [13] we see that $K_{-i}(\Gamma) = 0$ for $i \geqslant 2$ and all virtually nilpotent groups Γ. Using the spectral sequences in 3.3 and 3.7(a) (also [23]), we conclude that

$$K_{-i}(\Gamma)_c = K_{-i}(\Gamma)_f = 0, \quad \text{for all } i \geqslant 2.$$

Finally, we are going to show that the forgetful map $\tilde{K}_i(\Gamma)_f \to \tilde{K}_i(\Gamma)$ is a monomorphism, for Γ a virtually nilpotent group as in 3.1.1.

PROPOSITION 3.9. *The forgetful map $\tilde{K}_i(\Gamma)_f \to \tilde{K}_i(\Gamma)$ for $i \leqslant 1$ is a monomorphism for each fibering apparatus for Γ.*

Proof. We use the idea of the proof of Theorem 2.14 in [4]. It is enough to show the proposition for $i = 1$. We will use the Notation of 3.1.1. Let x be an element in the kernel of the forgetful map, and assume that x has diameter d in $M_{\hat{\Gamma}}/F$. Let $k : X \to M_\Gamma$ be a G-strong deformation retract representing x, where X is a finite G-CW complex. Then there is a finite G-CW complex, Z, and G-CEPL maps $\beta' : Z \to M_\Gamma$, $\beta : Z \to X$, such that $k\beta \simeq_G \beta'$, relM_Γ. Assume that the G-homotopy has diameter D in $M_{\hat{\Gamma}}/F$. Let g be an s-expansive map. Then $g^!_f(x)$ is represented by a d/s-G-strong deformation retract, and the lifting of the homotopy is a homotopy of diameter D/s, in $M_{\hat{\Gamma}}/F$. For s sufficiently large, $g^!_f(x) = 0 \in \mathrm{Wh}(\Gamma_g)_{gf'}$. By Proposition 3.7, $g^!_f$ is a monomorphism, and so $x = 0$.

4. On the Negative K-Groups of Cocompact Subgroups of Lie Groups

Let Γ be a subgroup of a cocompact subgroup of a virtually connected Lie group (i.e. a Lie group with finitely many components). Then one of the main results in [13], states that $K_{-1}(\mathbb{Z}\Gamma)$ is generated by the images of $K_{-1}(\mathbb{Z}L)$, where L is a finite subgroup of Γ. Also, the main vanishing result in [13], that $K_{-i}(\mathbb{Z}\Gamma) = 0$ for $i \geqslant 2$ An easy application of the Bass–Heller–Swan formula shows that the twisted lower Nil-groups vanish, i.e. $NK_{-i}(\mathbb{Z}\Gamma, \alpha) = 0$ for $i \geqslant 2$, for any automorphism α of Γ.

PROPOSITION 4.1. *Let Γ be as above then $NK_{-1}(\mathbb{Z}\Gamma, \alpha) = 0$.*

Proof. Notice that Γ and $\Gamma \times_\alpha \mathbb{Z}$ have isomorphic finite subgroups. So the inclusion induced map $K_{-1}(\mathbb{Z}\Gamma) \to K_{-1}(\mathbb{Z}[\Gamma \times_\alpha \mathbb{Z}])$ is an epimorphism. But from Bass–Heller–Swan [9] and the fact that $K_{-2}(\mathbb{Z}\Gamma) = 0$, we derive an exact sequence

$$K_{-1}(\mathbb{Z}\Gamma) \to K_{-1}(\mathbb{Z}[\Gamma \times_\alpha \mathbb{Z}]) \to NK_{-1}(\mathbb{Z}\Gamma, \alpha) \oplus NK_{-1}(\mathbb{Z}\Gamma, \alpha^{-1}) \to 0,$$

where the first map is induced by the inclusion, and so it is an epimorphism. Therefore the NK_{-1} groups vanish.

Let Γ be a virtually nilpotent group as in Chapter 3. We assume also that G has odd order. Using Proposition 4.1, we will show that the forget control map $K_{-1}(\Gamma)_c \to K_{-1}(\Gamma)$ is an isomorphism.

We start first with the case of crystallographic groups. Let $\hat{\Gamma}$ be a crystallographic group. Then there is an exact sequence $1 \to A \to \hat{\Gamma} \xrightarrow{p} F \to 1$, where A is free Abelian and F is a finite group acting faithfully on A. In our situation, we will assume that F has odd order. In [10] there is a classification of crystallographic groups of odd order holonomy:

(a) There is an epimorphism $\hat{\Gamma} \to \mathbb{Z}$ with kernel a crystallographic group.
(b) $\hat{\Gamma}$ satisfies hypothesis $\mathcal{H}$: there are infinitely many numbers, $s \equiv 1 \mathrm{mod} |F|$, such that any hyperelementary subgroup of $\hat{\Gamma}_s = \hat{\Gamma}/sA$ which projects to F, under the natural epimorphism, projects isomorphically to F.

The above classification splits the proofs of certain statements in this paper in three cases:

Case 1: When $\hat{\Gamma}$ satisfies (a).
Case 2: When $\hat{\Gamma}$ satisfies hypothesis $\mathcal{H}$ and F is not hyperelementary.
Case 3: When $\hat{\Gamma}$ satisfies hypothesis $\mathcal{H}$ and F is hyperelementary.

The above classification is useful because it provides tools for applying induction and hyperelementary induction in the proofs. In [4] it was shown that if $p : \hat{\Gamma} \to K$ is any group epimorphism to a finite group then the functors $\tilde{K}_i()$ and $\tilde{K}_i()_c$ satisfy hyperelementary induction

$$\tilde{K}_i(\hat{\Gamma}) \cong \lim_{\longleftarrow} \tilde{K}_i(p^{-1}(H)), \qquad \tilde{K}_i(\hat{\Gamma})_c \cong \lim_{\longleftarrow} \tilde{K}_i(p^{-1}(H))_c,$$

where the inverse limit is taken over the class of all hyperelementary subgroups of K and the isomorphism is induced by restriction maps.

PROPOSITION 4.2. *Let $\hat{\Gamma}$ be a crystallographic group with odd order holonomy. Then the forget control map $K_{-1}(\hat{\Gamma})_c \to K_{-1}(\hat{\Gamma})$ is an isomorphism.*

Proof. We use the methods developed in [4]. We will use induction first on $\mathrm{vcd}(\hat{\Gamma})$ and then on the order of the holonomy group.

Case 1. Assume that there is an epimorphism $\hat{\Gamma} \to \mathbb{Z}$. Then $\hat{\Gamma} \cong \Delta \times_\alpha \mathbb{Z}$, where Δ is a crystallographic group such that $\mathrm{vcd}(\Delta) = \mathrm{vcd}(\hat{\Gamma}) - 1$. So by the induction hypothesis $K_{-1}(\Delta)_c \cong K_{-1}(\Delta)$. Using Proposition 4.1 we get a commutative diagram

$$\begin{array}{ccccccc}
K_{-1}(\Delta)_c & \xrightarrow{1-\alpha_*} & K_{-1}(\Delta)_c & \longrightarrow & K_{-1}(\hat{\Gamma})_c & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K_{-1}(\Delta) & \xrightarrow{1-\alpha_*} & K_{-1}(\Delta) & \longrightarrow & K_{-1}(\hat{\Gamma}) & \longrightarrow & 0,
\end{array}$$

where the top horizontal exact sequence is the splitting given in [4], the bottom exact sequence is the classical Bass–Heller–Swan splitting in [9], and the vertical maps are the forget control maps. By assumption the first two vertical maps are isomorphisms, so the third vertical map is an isomorphism.

Case 2. Assume $\hat{\Gamma}$ satisfies hypothesis $\mathcal{H}$ and F is not a hyperelementary group. By the induction techniques developed in [4], the restriction maps induce an isomorphism $K_{-1}(\hat{\Gamma})_c \xrightarrow{\cong} \lim_{\longleftarrow} K_{-1}(\hat{\Gamma}_H)_c$, where $\hat{\Gamma}_H = p^{-1}(H)$ and the inverse limit is taken over the hyperelementary subgroups of F. There is a similar induction isomorphism for $K_{-1}(\hat{\Gamma})$ and the following diagram commutes

$$\begin{array}{ccc} K_{-1}(\hat{\Gamma})_c & \longrightarrow & \varprojlim K_{-1}(\hat{\Gamma}_H)_c \\ \downarrow & & \downarrow \\ K_{-1}(\hat{\Gamma}) & \longrightarrow & \varprojlim K_{-1}(\hat{\Gamma}_H). \end{array}$$

Notice that $\hat{\Gamma}_H$ is a crystallographic group with holonomy of order less than $|F|$. So the forget control map, for $\hat{\Gamma}_H$, is an isomorphism. Therefore the forget control map $K_{-1}(\hat{\Gamma})_c \to K_{-1}(\hat{\Gamma})$ is an isomorphism.

Case 3. Assume $\hat{\Gamma}$ satisfies condition $\mathcal{H}$ and F is hyperelementary. In [4] Theorem 2.14, it was shown that the forget control map is a monomorphism. We will identify $K_{-1}(\hat{\Gamma})_c$ with its image in $K_{-1}(\hat{\Gamma})$. Notice that for each $s \equiv 1 \mathrm{mod} |F|$, $H^k(F; A_s) = 0$ where $A_s = A/sA$ and therefore the exact sequence $1 \to A_s \to \hat{\Gamma}_s \to F \to 1$ splits and there is exactly one conjugacy class of subgroups of $\hat{\Gamma}_s$ which is sent isomorphically onto F by the natural epimorphism $\hat{\Gamma}_s \to F$ ([12]). Since $\hat{\Gamma}$ satisfies condition $\mathcal{H}$, for infinitely many $s \equiv 1\mathrm{mod}|F|$, that is the unique class of hyperelementary subgroups of $\hat{\Gamma}_s$ whose order equal to the order of F. Let K be a subgroup in this class. If H is any hyperelementary subgroup of $\hat{\Gamma}_s$, not conjugate to K, then the holonomy group of $\hat{\Gamma}_H$ is isomorphic to a proper subgroup of F and consequently the holonomy number of Γ_H is strictly less than the holonomy number of Γ. Also there is an s-expansive map $g : \hat{\Gamma} \to \hat{\Gamma}$ such that $g(\hat{\Gamma}) = \hat{\Gamma}_K$.

Let x be an element of $K_{-1}(\hat{\Gamma})$. By the stability of the controlled groups, there is an integer S such that: for each $s > S$, $s \equiv 1\mathrm{mod}|F|$, and each s-expansive map g, $g^!(x) \in K_{-1}(\hat{\Gamma})_c$.

If we choose s as above then for each hyperelementary subgroup H of $\hat{\Gamma}_s$, not conjugate to K, $\mathrm{res}_H^{\hat{\Gamma}_s}(x) \in K_{-1}(\hat{\Gamma}_H)$ and by the induction hypothesis $\mathrm{res}_H^{\hat{\Gamma}_s}(x) \in K_{-1}(\hat{\Gamma}_H)_c$. Also, $\mathrm{res}_K^{\hat{\Gamma}_s} = g^!$ which implies that $\mathrm{res}_K^{\hat{\Gamma}_s}(x) = g^!(x) \in K_{-1}(\hat{\Gamma}_K)_c$. Thus, if $\mathrm{res} : K_{-1}(\hat{\Gamma}) \to \lim_{\longleftarrow} K_{-1}(\hat{\Gamma}_H)$ is the restriction map $\mathrm{res}(x) \in \lim_{\longleftarrow} K_{-1}(\hat{\Gamma}_H)_c$. Therefore $x \in K_{-1}(\hat{\Gamma})_c$ which completes the proof in Case 3 and the proof of the proposition.

By applying Proposition 4.2 to the other components of the direct sum decomposition of $K^{PL}_{-1,F}(M_{\hat{\Gamma}})$ we derive

COROLLARY 4.3. *The forget control map*

$$K^{\mathrm{PL}}_{-1,F}(M_{\hat{\Gamma}})_c \to K^{\mathrm{PL}}_{-1,F}(M_{\hat{\Gamma}})$$

is an isomorphism. Therefore $K^{\mathrm{Top}}_{-1,F}(M_{\hat{\Gamma}}) = 0$.

Let Γ be a virtually nilpotent group such that G has odd order. We are going to prove the analogue of Proposition 4.2 for Γ. Define the holonomy number of Γ, $h(\Gamma)$, to be the minimum of the orders of the holonomy groups of the crystallographic groups appearing in a fibering apparatus of Γ. Let $\mathcal{A} = (\hat{\Gamma}, \phi, f)$ be a fibering apparatus for Γ as in 3.1.1. Let $q : \hat{\Gamma} \to K$ be an epimorphism to the holonomy group F, or one of the finite groups $\hat{\Gamma}_s$. Then the methods of [4] show that the functor $\tilde{K}_i()_f$ satisfies hyperelementary induction $\tilde{K}_i(\Gamma)_f \cong \lim_{\longleftarrow} \tilde{K}_i(\Gamma_H)_f$, where $\Gamma_H = \phi^{-1}q^{-1}(H)$, the limit is taken over the hyperelementary subgroups of K and the isomorphism is induced by the restriction maps. Notice that Γ_H is a virtually nilpotent group with the same virtual cohomological dimension as Γ and $\mathcal{A}_H = (q^{-1}(H), \phi|, f)$ is a fibering apparatus for Γ_H.

THEOREM 4.4. *Let Γ be as above. Then the forget control map $K_{-1}(\Gamma)_c \to K_{-1}(\Gamma)$ is an isomorphism.*

Proof. We are going to use induction first on the $\mathrm{vcd}(\Gamma)$ and then on $h(\Gamma)$. Let $\mathcal{A} = (\hat{\Gamma}, \phi, f)$ be a fibering apparatus for Γ as in 3.1.1.

Case 1. Assume that there is an epimorphism $\hat{\Gamma} \to \mathbb{Z}$. Then there is an epimorphism $\Gamma \to \mathbb{Z}$ with kernel a virtually nilpotent group with virtual cohomological dimension less than $\mathrm{vcd}(\Gamma)$. The proof can be completed as in Case 1, Proposition 4.2. Notice also that the forget control map factors through $K_{-1}(\Gamma)_f$ and the map $K_{-1}(\Gamma)_f \to K_{-1}(\Gamma)$ is a monomorphism (Proposition 3.9). Therefore the map $f_\# : K_{-1}(\Gamma)_c \to K_{-1}(\Gamma)_f$ is an isomorphism.

Case 2. Assume that $\hat{\Gamma}$ satisfies condition $\mathcal{H}$ and F is not hyperelementary. Then the proof can be completed as in Case 2, Proposition 4.2. Notice that the map $f_\# : K_{-1}(\Gamma)_c \to K_{-1}(\Gamma)_f$ is an isomorphism in this case.

Case 3. Assume that $\hat{\Gamma}$ satisfies condition $\mathcal{H}$ and F is a hyperelementary group. First we will show that the forget control map $K_{-1}(\Gamma)_f \to K_{-1}(\Gamma)$ is an isomorphism. We use the same argument as in Proposition 4.2. For infinitely many s, $s \equiv 1 \mathrm{mod} |F|$, there is a unique conjugacy class of hyperelementary subgroups of $\hat{\Gamma}_s$ which project isomorphically to F. Let K be a group in this class. For $x \in K_{-1}(\Gamma)$, and for each hyperelementary subgroup H of $\hat{\Gamma}_s$, not conjugate to K $\mathrm{res}_H^{\hat{\Gamma}_s}(x) \in K_{-1}(\Gamma_H)_f$ by induction, because $h(\Gamma_H) < h(\Gamma)$. Also, $\mathrm{res}_K^{\hat{\Gamma}_s} = g^!$, for some s-expansive map g. So for large enough s $\mathrm{res}_K^{\hat{\Gamma}_s}(x) = g^!(x) \in K_{-1}(\Gamma_K)_f$. In every case

$$\mathrm{res}(x) \in \lim_{\longleftarrow} K_{-1}(\Gamma_H)_f \cong K_{-1}(\Gamma)_f.$$

Therefore $x \in K_{-1}(\Gamma)_f$, the forget control map is an epimorphism and $K_{-1}(\Gamma)_f \cong K_{-1}(\Gamma)$. Using Remark 3.8, we see that

$$K_{-1}(\Gamma)_c \cong H_0(M_{\hat{\Gamma}}/F, K_{-1}(\Gamma_v)_c) \cong H_0(M_{\hat{\Gamma}}/F, K_{-1}(\Gamma_v)) \cong K_{-1}(\Gamma)_f,$$

where the second isomorphism is obtained using the induction hypothesis, since $\mathrm{vcd}(\Gamma_v) < \mathrm{vcd}(\Gamma)$. Therefore $K_{-1}(\Gamma)_c \cong K_{-1}(\Gamma)$.

COROLLARY 4.5. *The forget control map* $K^{\mathrm{PL}}_{-1,G}(M_\Gamma)_c \to K^{\mathrm{PL}}_{-1,G}(M_\Gamma)$ *is an isomorphism. Therefore* $K^{\mathrm{Top}}_{-1,G}(M_\Gamma) = 0$.

5. A Split Short Exact Sequence

In this chapter we will show that the five term exact sequence introduced by Steinberger and West [21], [22], reduces, in the case of a finite group of odd order acting by isometries on a nilmanifold, to split short exact sequences.

Let Γ be as in $(*)$ together with a fibering apparatus $\mathcal{A} = (\hat{\Gamma}, \phi, f)$ as in 3.1.1. We assume that G is an odd order group which implies that the holonomy group F has odd order. We are going to show that the forget control map $\tilde{K}_i(\Gamma)_c \to \tilde{K}_i(\Gamma)$, for $i \leqslant 1$, is a split monomorphism. The proof will be done in two stages. First we will show that the forget control map $\tilde{K}_i(\Gamma)_f \to \tilde{K}_i(\Gamma)$ is a split monomorphism using the methods developed in [4]. Then we will show that the forgetful map $\tilde{K}_i(\Gamma)_c \to \tilde{K}_i(\Gamma)_f$ is a split monomorphism.

Let $N_1(\Gamma)$ be the subgroup of $\mathrm{Wh}(\Gamma)$ consisting of those elements that vanish under the transfer map of an s-expansive monomorphism, $s \equiv 1 \mathrm{mod}|F|$, for infinitely many numbers s i.e. $N_1(\Gamma)$ consists of those elements of $\mathrm{Wh}(\Gamma)$ that lie in the zero eigenspace of s-expansive maps for infinitely many $s \equiv 1 \mathrm{mod}|F|$. We also define $N_i(\Gamma)$, $i < 1$, as the set of elements in the zero eigenspace of all maps on $\tilde{K}_i(\Gamma)$ induced by s-expansive maps, for infinitely many $s \equiv 1 \mathrm{mod}|F|$. We will show that the forget control map $\tilde{K}_i(\Gamma)_f \to \tilde{K}_i(\Gamma)$ is a split monomorphism and the orthogonal summand of $\tilde{K}_i(\Gamma)_f$ is $N_i(\Gamma)$.

We start by proving an analogue of Bass–Heller–Swan formula for $\tilde{K}_i(\Gamma)_f$. Let $\mathcal{A} = (\hat{\Gamma}, \phi, f)$ be a fibering apparatus for Γ as in 3.1.1. Assume that there is a group epimorphism $\psi : \hat{\Gamma} \to \mathbb{Z}$, with kernel a crystallographic group $\hat{\Gamma}'$ of rank $m-1$. Then $\hat{\Gamma} = \hat{\Gamma}' \times_\beta \mathbb{Z}$. The map ψ induces a fiber bundle $\mathbb{R}^m/\hat{\Gamma} \to S^1$ with fiber $\mathbb{R}^{m-1}/\hat{\Gamma}'$, [26]. Also ψ induces an epimorphism $\Gamma \to \mathbb{Z}$ with kernel a virtually nilpotent group $\Gamma' = \phi^{-1}(\hat{\Gamma}')$. Then $\Gamma = \Gamma' \times_\alpha \mathbb{Z}$. The map f induces a fiber bundle $\mathbb{R}^n/\Gamma \to S^1$ with fiber $\mathbb{R}^{n-1}/\Gamma'$. Then $\mathcal{A}' = (\hat{\Gamma}', \phi|, f|)$ is a fibering apparatus for Γ'. Set $f_1 = f|$.

LEMMA 5.1. *With the preceding notation, there is an exact sequence*

$$\tilde{K}_i(\Gamma')_{f_1} \xrightarrow{1-\alpha_*} \tilde{K}_i(\Gamma')_{f_1} \to \tilde{K}_i(\Gamma)_f \to \tilde{K}_{i-1}(\Gamma')_{f_1} \xrightarrow{1-\alpha_*} \tilde{K}_{i-1}(\Gamma')_{f_1}.$$

Proof. Notice that there is a bundle $M_\Gamma \to S^1$ with fiber $M_{\Gamma'}$. The controlled K-groups satisfy the Mayer–Vietoris property. This follows from [25] and their definition as generalized homology theory ([26], Proposition 2.1). The exact sequence now follows as in Lemma 3.5 in [4].

The exact sequence in 5.1 can be written as a short exact sequence

$$0 \to (\tilde{K}_i(\Gamma')_{f_1})_\alpha \to \tilde{K}_i(\Gamma)_f \to (\tilde{K}_{i-1}(\Gamma')_{f_1})^\alpha \to 0,$$

where $(\tilde{K}_i(\Gamma')_{f_1})_\alpha = \ker(1-\alpha_*)$ and $(\tilde{K}_{i-1}(\Gamma')_{f_1})^\alpha = \operatorname{coker}(1-\alpha_*)$.

The next proposition uses methods similar to the ones used in [4], Section 3.

PROPOSITION 5.2. *The forgetful map is a split monomorphism, and induces a direct sum decomposition* $\tilde{K}_i(\Gamma) \cong \tilde{K}_i(\Gamma)_f \oplus N_i(\Gamma)$, $i \leqslant 1$.

Proof. First we will show that the two subgroups are orthogonal. Let $x \in \tilde{K}_i(\Gamma)_f \cap N_i(\Gamma)$. Then for some s, $s \equiv 1 \bmod |F|$, and some s-expansive map g, $g^!(x) = 0$. Since $x \in \tilde{K}_i(\Gamma)_f$, $g^!_f(x) = 0$. But by Proposition 3.7, $g^!_f$ is a monomorphism on $\tilde{K}_i(\Gamma)_f$. Therefore $x = 0$. Notice that $\tilde{K}_i(\Gamma)_f \oplus N_i(\Gamma)$ is a subgroup of $\tilde{K}_i(\Gamma)$. We complete the proof by showing the other inclusion. We will use induction first on $\operatorname{vcd}(\Gamma)$ and then on the holonomy number of Γ, $h(\Gamma)$. For this we use the classification of crystallographic groups with odd order holonomy, as in the proof of Theorem 4.4. Let $\mathcal{A} = (\hat{\Gamma}, \phi, f)$ be a fibering apparatus for Γ.

Case 1. Assume that there is an epimorphism $\hat{\Gamma} \to \mathbb{Z}$. Then by Lemma 5.1 and the results of Farrell–Hsiang ([9]) we get a commutative diagram (also [4], 3.6)

$$\begin{tikzcd}
0 \arrow[r] & (\tilde{K}_i(\Gamma')_{f_1})_\alpha \arrow[r] \arrow[d] & \tilde{K}_i(\Gamma)_f \arrow[r] \arrow[d] & (\tilde{K}_{i-1}(\Gamma')_{f_1})^\alpha \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\tilde{K}_i(\Gamma'))_\alpha \arrow[r] & \tilde{K}_i(\Gamma)/C_i \arrow[r] & (\tilde{K}_{i-1}(\Gamma'))^\alpha \arrow[r] & 0
\end{tikzcd}$$

where C_i denotes the summand corresponding to the exotic Nil-groups. The first and the third vertical maps are monomorphisms, therefore the second map is a monomorphism. An s-expansive map g induces maps of the above exact sequences and thus of the exact sequence of the quotients

$$0 \to (N_i(\Gamma'))_\alpha \to \frac{\tilde{K}_i(\Gamma)}{C_i} \oplus \tilde{K}_i(\Gamma)_f \to (N_{i-1}(\Gamma'))^\alpha \to 0.$$

The summand $C_i \subset N_i(\Gamma)$ because the elements of C_i vanish after transfers along the last $\mathbb{Z}$-direction. From the above exact sequence we get an exact sequence of the zero eigenspaces of the expansive maps

$$0 \to (N_i(\Gamma'))_\alpha \to \frac{N_i(\Gamma)}{C_i} \to (N_{i-1}(\Gamma'))^\alpha \to 0,$$

which implies that $\tilde{K}_i(\Gamma) \subset \tilde{K}_i(\Gamma)_f \oplus N_i(\Gamma)$. This completes the proof in Case 1.

Case 2. Assume that $\hat{\Gamma}$ satisfies hypothesis $\mathcal{H}$ and F is not hyperelementary. The restriction maps induce isomorphisms

$$\tilde{K}_i(\Gamma) \cong \varprojlim \tilde{K}_i(\phi^{-1}q^{-1}(H)) \quad \text{and} \quad \tilde{K}_i(\Gamma)_f \cong \varprojlim \tilde{K}_i(\phi^{-1}q^{-1}(H))_f.$$

The first isomorphism commutes with transfer maps induced by s-expansive maps. Therefore it induces an isomorphism $N_i(\Gamma) \cong \lim_{\longleftarrow} N_i(\phi^{-1}q^{-1}(H))$, where the inverse limit is taken over the hyperelementary subgroups of F. Since inverse limits preserve direct sums and monomorphisms we get $\tilde{K}_i(\Gamma) \cong \tilde{K}_i(\Gamma)_f \oplus N_i(\Gamma)$.

Case 3. Assume that $\hat{\Gamma}$ satisfies hypothesis $\mathcal{H}$ and F is a hyperelementary group. The proof in this case is similar to the proof in Case 3 in Theorem 4.4.

Remark 5.2.1. The forgetful map $\iota_i : \tilde{K}_i(\Gamma)_c \to \tilde{K}_i(\Gamma)$ fuctors through $\tilde{K}_i(\Gamma)_f$. Therefore $\mathrm{Im}(\iota_i) \cap N_i(\Gamma) = \{0\}$ and the map $p_i : \tilde{K}_i(\Gamma) \to \tilde{K}_i^{\mathrm{Top}}(\Gamma)$ is a monomorphism when restricted to $N_i(\Gamma)$. We write $N_i'(\Gamma) = \tilde{K}_i^{\mathrm{Top}}(\Gamma)/N_i(\Gamma)$. Then the exact sequence

$$\tilde{K}_i(\Gamma)_c \to \tilde{K}_i(\Gamma) \to \tilde{K}_i^{\mathrm{Top}}(\Gamma) \to \tilde{K}_{i-1}(\Gamma)_c \to \tilde{K}_{i-1}(\Gamma)$$

induces the exact sequence, for $i \leqslant 1$,

$$\tilde{K}_i(\Gamma)_c \xrightarrow{f_\#} \tilde{K}_i(\Gamma)_f \to N_i'(\Gamma) \to \tilde{K}_{i-1}(\Gamma)_c \xrightarrow{f_\#} \tilde{K}_{i-1}(\Gamma)_f.$$

Next we study the forgetful map $f_\# : \tilde{K}_i(\Gamma)_c \to \tilde{K}_i(\Gamma)_f$. We will show that it is a split monomorphism.

PROPOSITION 5.3. *With the above notation, the forgetful map* $f_\# : \tilde{K}_i(\Gamma)_c \to \tilde{K}_i(\Gamma)_f$ *is a split monomorphism.*

Proof. We will use induction on $\mathrm{vcd}(\Gamma)$. Therefore the forget control map

$$h_\# : \tilde{K}_i(\Delta)_c \to \tilde{K}_i(\Delta)_h$$

is a split monomorphism if $\mathrm{vcd}(\Delta) < \mathrm{vcd}(\Gamma)$ (here h is a map appearing in a fibering apparatus for Δ). Thus by Proposition 5.2, the forget control map $\tilde{K}_i(\Delta)_c \to \tilde{K}_i(\Delta)$ is a split monomorphism. By 3.7 there is a spectral sequence converging to $\tilde{K}_{i+j}(\Gamma)_f$ (for $i + j \leqslant 1$) with

$$E^2_{i,j} = H_i(M_{\hat{\Gamma}}/F, \tilde{K}_j(\Gamma_v)).$$

But $\mathrm{vcd}(\Gamma_v) < \mathrm{vcd}(\Gamma)$ and the assumption hypothesis implies that

$$\tilde{K}_j(\Gamma_v) = \tilde{K}_j(\Gamma_v)_c \oplus \tilde{K}_j^{\mathrm{Top}}(\Gamma_v), \quad i \leqslant 1.$$

The map $\tilde{K}_j(\Gamma_v)_c \to \tilde{K}_j(\Gamma_v)$ is the forget control map. Therefore the E^2-term above splits as a direct sum with one summand $H_i(M_{\hat{\Gamma}}/F, \tilde{K}_j(\Gamma_v)_c)$ which is the

E^2-term of the spectral sequence of 3.3 converging to $\tilde{K}_{i+j}(\Gamma)_c$. By Lemma 3.8 the forget control map induces a map

$$E'^2_{i,j} = H_i(M_{\hat{\Gamma}}/F; \tilde{K}_j(\Gamma_v)_c) \xrightarrow{A'_{ij}} H_i(M_{\hat{\Gamma}}/F; \tilde{K}_j(\Gamma_v)) = E^2_{i,j},$$

on the E^2-terms of the spectral sequences of Proposition 3.3 and Proposition 3.7 that in turn induces the map $\bar{f}_\# : \tilde{K}_i(\Gamma)_c \to \tilde{K}_i(\Gamma)_f$ on the associated graded groups. Since $\mathrm{vcd}(\Gamma_v) < \mathrm{vcd}(\Gamma)$, the induction hypotheses implies that the forget control map

$$(f_v)_\# : \tilde{K}_j(\Gamma_v)_c \to \tilde{K}_j(\Gamma_v)_{f_v}, \quad j \leqslant 1$$

is a split monomorphism. Thus the five-term exact sequence of Remark 5.2.1 is reduced to a split short exact sequence

$$0 \to \tilde{K}_j(\Gamma_v)_c \xrightarrow{(f_v)_\#} \tilde{K}_j(\Gamma_v)_{f_v} \to N'_j(\Gamma_v) \to 0, \quad j \leqslant 1.$$

The proof that $f_\#$ is a split monomorphism will be done in two steps.

Step 1. The map $f_\#$ is a monomorphism and there is an exact sequence

$$0 \to \tilde{K}_j(\Gamma)_c \xrightarrow{f_\#} \tilde{K}_j(\Gamma)_f \to N'_j(\Gamma) \to 0, \quad j \leqslant 1.$$

Proof. The map A'_{ij} on the E^2-terms is a split monomorphism. Therefore the map $\bar{f}_\#$ induced on the E^∞-terms is a split monomorphism of graded groups. A diagram chase shows that $f_\#$ is a monomorphism, not necessarily split. The short exact sequence is derived from the five-term exact sequence of Remark 5.2.1.

Step 2. The map $f_\#$ is a split monomorphism.

Proof. The group $E^2_{i,j}$ splits as a direct sum for $j \leqslant 1$

$$H_i(M_{\hat{\Gamma}}/F, \tilde{K}_j(\Gamma_v)_c) \oplus H_i(M_{\hat{\Gamma}}/F, N'_j(\Gamma_v)).$$

The inclusion and projection maps

$$\tilde{K}_j(\Gamma_v)_c \to \tilde{K}_j(\Gamma_v) \to \tilde{K}_j(\Gamma_v)_c$$

induce maps in the E^2-terms of the two spectral sequences. The maps commute with the differentials. Thus d_2 is given as a diagonal matrix

$$\begin{pmatrix} d'_2 & 0 \\ 0 & d''_2 \end{pmatrix} : H_i(M_{\hat{\Gamma}}/F, \tilde{K}_j(\Gamma_v)_c) \oplus H_i(M_{\hat{\Gamma}}/F, N'_j(\Gamma_v))$$

$$\to H_{i-2}(M_{\hat{\Gamma}}/F, \tilde{K}_{j+1}(\Gamma_v)_c) \oplus H_{i-2}(M_{\hat{\Gamma}}/F, N'_{j+1}(\Gamma_v)),$$

for $j \leqslant 0$. Then in the limit of the spectral sequences we get an isomorphism

$$\tilde{K}_{i+j}(\Gamma)_c \oplus N'_{i+j}(\Gamma) \to \tilde{K}_{i+j}(\Gamma)_f,$$

where the first map is $f_\#$ (the supplementary summand of $\tilde{K}_{i+j}(\Gamma)$ is $N'_{i+j}(\Gamma)$ being the cokernel of $f_\#$). Therefore the forget control map $f_\#$ is a split monomorphism.

By combining the results in Propositions 5.2, 5.3, we derive the following

THEOREM 5.4. *With the above notation*

(a) *The forget control map* $\tilde{K}_i(\Gamma)_c \to \tilde{K}_i(\Gamma)$ *is an isomorphism for* $i \leqslant -1$ *and a split monomorphism for* $i = 0, 1$.
(b) *The forget control map* $\tilde{K}^{\mathrm{PL}}_{i,G}(M_\Gamma)_c \to \tilde{K}^{\mathrm{PL}}_{i,G}(M_\Gamma)$ *is an isomorphism for* $i \leqslant -1$ *and a split monomorphism for* $i = 0, 1$.
(c) *The sequence*

$$0 \to \tilde{K}^{\mathrm{PL}}_{i,G}(M_\Gamma)_c \to \tilde{K}^{\mathrm{PL}}_{i,G}(M_\Gamma) \to \tilde{K}^{\mathrm{Top}}_{i,G}(M_\Gamma) \to 0$$

is split exact for $i \leqslant 1$.

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