Spectral Theory of Graphs

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Basic Definitions

Let G be a graph with $V(G)$ the set of vertices and $E(G)$ the set of edges. We assume that each edge has two endpoints. Here are some standard definitions.

Definition

1. Two vertices v, u contained in $V(G)$ are adjacent, denoted $u \sim v$, if they are endpoints of the same edge. For adjacent vertices, write (u, v) for the edge they determine.
2. For $u \in V$,
   $$N(v) = \{u \in V(G);\ v \sim u\}.$$
   Elements of $N(v)$ are called **neighbours** of v. The neighbourhood is the entire set $N(v)$.
3. A vertex v and an edge e are incident if v is an endpoint of e.

In many applications the edges of a graph naturally come with a weight.

Definition

1. A **weighted graph** G is a graph with vertex set $V(G)$, edge set $E(G)$, and a weight function defined as follows:

$$w : V(G) \times V(G) \to [0, \infty)$$

   where

   1. $w(u, v) = 0$ when (u, v) is not an edge,
   2. $w(u, v) > 0$ if (u, v) is an edge.

We define the degree of a vertex v:

$$deg(v) = \sum_{u \in V(G)} w(v, u).$$

Remark

1. We will also assume that vertices u, v are adjacent if $w(u, v) > 0$.
2. If all the edges of a weighted graph have weight 1 we will call it an **unweighted graph**; so

$$w(u, v) = \begin{cases} 1, & \text{if } u \sim v, \\ 0, & \text{if } u \text{ is not adjacent to } v. \end{cases}$$

Finite Graphs

First we assume that $|V(G)| = n < \infty$. The adjacency matrix, denoted by $A(G)$, of a weighted finite graph is defined as

$$A(G)_{v,u} = \begin{cases} w(v,u), & v \sim u \\ 0, & \text{otherwise} \end{cases}$$

The set of the eigenvalues of $A(G)$ is called the spectrum of G,

Definition

A k-**regular graph** is an unweighted graph where all the vertices have the same degree k. A regular graph is a k-regular graph for some k.

Laplacians

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. The Laplacian of f is defined as:

$$
\begin{aligned}
\mathcal{L}(f) &= \frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} \\
&= \left[\frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_n}\right] \begin{bmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \cdots \\ \frac{\partial}{\partial x_n} \end{bmatrix} f
\end{aligned}
$$

Notice that $\mathcal{L}$ can be written as:

$$\mathcal{L}(f) = \nabla \nabla^T (f)$$

where ∇ is the gradient.

In a directed graph G we define the **incidence matrix** to be a $|E(G)| \times |V(G)|$-matrix as

$$\begin{array}{rcll} \nabla_{ev} & = & -1 & \text{if } v \text{ is the initial vertex of } e \\ \nabla_{ev} & = & 1 & \text{if } v \text{ is the terminal vertex of } e \\ \nabla_{ev} & = & 0 & \text{if } v \text{ is not a vertex of } e \end{array}$$

and the combinatorial Laplacian is defined as $L = \nabla^T \nabla$.

For a graph G on n vertices, the Combinatorial Laplacian is defined

$$L(u,v) = \begin{cases} d_v - w(v,v), & \text{if } u = v, \\ -w(u,v), & \text{if} u \sim v, \\ 0, & \text{otherwise.} \end{cases}$$

Remark

For unweighted graphs, the combinatorial laplacian is

$$L(u,v) = \begin{cases} d_v, & \text{if } u = v, \\ -1, & \text{if} u \sim v, \\ 0, & \text{otherwise.} \end{cases}$$

The **Normalised Laplacian** of weighted graphs is denoted as $\mathcal{L}(u,v)$ where

$$\mathcal{L}(u,v) = \begin{cases} 1 - \dfrac{w(v,v)}{d_v}, & \text{if } u = v, \\ -\dfrac{w(u,v)}{\sqrt{d_u d_v}}, & \text{if} u \sim v, \\ 0, & \text{otherwise.} \end{cases}$$

Remark

1. For unweighted graphs the normalised Laplacian is given by

$$\mathcal{L}(u,v) = \begin{cases} 1, & \text{if } u = v, \\ -\dfrac{1}{\sqrt{d_u d_v}}, & \text{if} u \sim v, \\ 0, & \text{otherwise.} \end{cases}$$

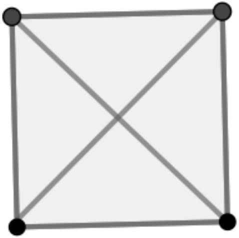
2. For k-regular graphs,

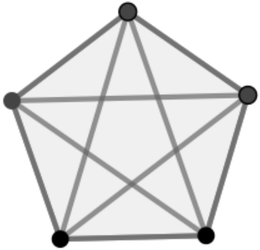
$$\mathcal{L} = I - \frac{1}{k} A$$

where A is the adjacency matrix.

Example 1 - The Complete Graph

The Complete Graph. let K_n be the complete graph in n vertices which is $n-1$-regular.





Then the adjacency matrix $A_n = J_n - I_n$ where J_n is the matrix with all entries equal to 1 and I_n is the identity matrix. The characteristic polynomial is given as $\psi_n(x) = \det((x+1)I_n - J_n)$. Lets calculate

$$\psi'_n(x) = \det(xI_n - J_n) = \begin{vmatrix} x-1 & -1 & -1 & \dots & -1 \\ -1 & x-1 & -1 & \dots & -1 \\ \ddots & \ddots & \ddots & & \ddots \\ -1 & -1 & -1 & \dots & x-1 \end{vmatrix}$$

Adding all the columns to the first column:

$$\begin{vmatrix} x-n & -1 & -1 & \dots & -1 \\ x-n & x-1 & -1 & \dots & -1 \\ \ddots & \ddots & \ddots & & \ddots \\ x-n & -1 & -1 & \dots & x-1 \end{vmatrix} = (x-n) \begin{vmatrix} 1 & -1 & -1 & \dots & -1 \\ 1 & x-1 & -1 & \dots & -1 \\ \ddots & \ddots & \ddots & & \ddots \\ 1 & -1 & -1 & \dots & x-1 \end{vmatrix}$$

Adding the first column to the other columns

$$\psi_n'(x) = (x-n)\begin{vmatrix} 1 & 0 & 0 & \dots & \\ 1 & x & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & & \ddots \\ 1 & 0 & 0 & \dots & x \end{vmatrix} = (x-n)x^{n-1}$$

So the roots are n with multiplicity 1 and 0 with multiplicity $n-1$. Thus the eigevalues of A are $n-1$ with multiplicity 1 and -1 with multiplicity $n-1$.

Example 2 - The Circle

The circle of n points. Let C_n be the 2-regular graph on n vertices (a circle with n points on it). We notice that

$$A_n = P + P^T = P + P^{-1}$$

where

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & & \ddots \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

The matrix P is called a permutation matrix because it corresponds to permuting the columns of I_n according to the permutation $\sigma = (12 \dots n)$. Since σ has order n in the symmetric group on n letters, P has oder n i.e. $P^n = I_n$. Thus the eigenvalues of P are the n-th roots of unity $e^{\frac{2\pi ik}{n}}$, $k = 0, 1, \dots n-1$. The eigenvectors of P and P^{-1} coincide. Let v be an eigenvector of P (and P^{-1}) with eigenvalue λ, then

$$(\lambda + \lambda^{-1})v = \lambda v + \lambda^{-1} v = Pv + P^{-1}v = (P + P^{-1})v.$$

Therefore $\lambda + \lambda^{-1}$ is an eigenvalue of A_n. Thus the spectrum of C_n is

$$\left\{ 2\cos\left(\frac{2\pi k}{n}\right) : \ k = 0, 1, \dots n-1 \right\}$$

Example 3 - The Path

The path of n points. let P_n denotes the path with n vertices. The adjacency matrix is

$$A_n = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 1 & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}$$

To characterise the eigenvalues of P_n we introduce the Chebyshev polkynomials.

Chebyshev Polynomial

Definition

The Chebyshev polynomials of second type are defined inductively as:

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

$$\begin{aligned} U_2(x) &= 4x^2 - 1, \ U_3(x) = 8x^3 - 4x, \ U_4(x) = 16x^4 - 12x^2 - 1, \\ U_5(x) &= 32x^5 - 32x^3 + 6x, U_6(x) = 64x^6 - 80x^4 + 24x^2 - 1 \end{aligned}$$

Some of their properties: The generating function is:

$$\sum_{n=0}^{\infty} U_n(x)t^n = \frac{1}{1 - 2tx + t^2}$$

- The exponential function is:

$$\sum_{n=0}^{\infty} U_n(x)\frac{t^n}{n!} = e^{tx}\left(\cosh(t\sqrt{x^2-1}) + \frac{x}{\sqrt{x^2-1}}\sinh(t\sqrt{x^2-1})\right)$$

- Using trigonometric functions we can write:

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin}$$

- The roots of $U_n(x)$ are:

$$x_k = \cos\left(\frac{k}{n+1}\pi\right), \quad k = 1, 2, \ldots n.$$

For the characteristic polynomial of P_n we have

$$\chi_n(x) = \det(xI_n - A_n) = \begin{vmatrix} x & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & x & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & x & -1 & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & & \ddots & \\ 0 & 0 & 0 & 0 & \dots & -1 & x \end{vmatrix}$$

We expand using the first column:

$$\chi_n(x) = x \begin{vmatrix} x & -1 & 0 & \dots & 0 & 0 \\ -1 & x & -1 & \dots & 0 & 0 \\ \ddots & \ddots & \ddots & & \ddots & \\ 0 & 0 & 0 & \dots & -1 & x \end{vmatrix} + \begin{vmatrix} -1 & 0 & \dots & 0 & 0 \\ -1 & x & \dots & 0 & \\ \ddots & \ddots & & \ddots & \\ 0 & 0 & \dots & -1 & x \end{vmatrix}$$

Continuing with the second determinant:

$$\chi_n(x) = x\chi_{n-1}(x) - \begin{vmatrix} x & -1 & 0 & \dots & 0 \\ -1 & x & -1 & \dots & 0 \\ \ddots & \ddots & \ddots & & \ddots \\ 0 & 0 & 0 & \dots & x \end{vmatrix} = x\chi_{n-1}(x) - \chi_{n-2}(x),$$

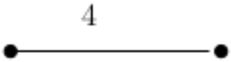
Thus

$$\chi_n(x) = U_n\left(\frac{x}{2}\right)$$

and the spectrum of P_n is

$$2\cos\left(\frac{k}{n+1}\pi\right), \quad k = 1, 2, \ldots n.$$

Example 4 - The line weighted graph



$$A = \begin{bmatrix} 0 & 4 \\ 4 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & -4 \\ -4 & 1 \end{bmatrix}$$

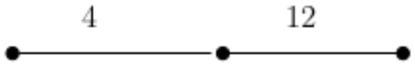
The normalized Laplacian is:

$$\mathcal{L} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

with eigenvalues 0 and 2 and characteristic polynomial $x^2 - 2x$.

Example 4 - The line weighted graph 2

The next line weighted graph:



$$A = \begin{bmatrix} 0 & 4 & 0 \\ 4 & 0 & 12 \\ 0 & 12 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & -4 & 0 \\ -4 & 2 & -12 \\ 0 & -12 & 1 \end{bmatrix}$$

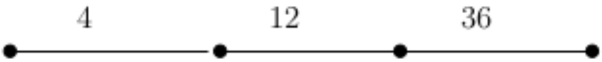
The normalised Laplacian is:

$$\mathcal{L} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 1 & -\frac{\sqrt{3}}{2} \\ 0 & -\frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$

with characteristic polynomial $x^3 - 3x^2 + 2x$ and eigenvalues 0, 1 and 2.

Example 4 - The line weighted graph 3

The next line weighted graph:



$$A = \begin{bmatrix} 0 & 4 & 0 & 0 & \\ 4 & 0 & 12 & 0 & \\ 0 & 12 & 0 & 36 & \\ 0 & 0 & & 36 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & -4 & 0 & 0 \\ -4 & 2 & -12 & 0 \\ 0 & -12 & 2 & -36 \\ 0 & 0 & -36 & 1 \end{bmatrix}$$

$$\mathcal{L} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 1 & -\frac{\sqrt{3}}{4} & 0 \\ 0 & -\frac{\sqrt{3}}{4} & 1 & -\frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & 1 \end{bmatrix}$$

with characteristic polynomial $x^4 - 4x^3 + \frac{77}{16}x^2 - \frac{13}{8}x$ and eigenvalues: $0, 2, 1 + \frac{\sqrt{3}}{4}, 1 - \frac{\sqrt{3}}{4}$

Some properties of the adjacency matrix and its spectrum

An edge path in a graph is a sequence of vertices $\tau = \{v_0, v_1, \dots v_m\}$ so that $(v_i, v_{i+1}$ is an edge. The number m is called the length of τ. A graph is connected if, for each two vertices v, u, there is an edge path from v to u. A maximal connected subgraph of a graph is called a connected component of a graph.

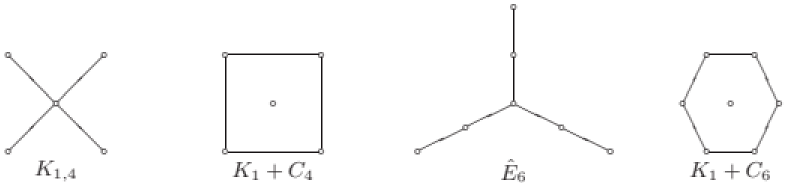
Proposition

The number of paths of length m from the vertex i to the vertex j is the entry $(A^m)_{ij}$.

1. The spectrum of G is the disjoint union of the spectra of its components.
2. The multiplicity of 0 as a Laplace eigenvalue of an undirected graph G equals the number of connected components of G.
3. Let the undirected graph k-regular graph G, k is the largest eigenvalue of G and its multiplicity equals the number of connected components of G.

Two graphs (V_i, E_i), $i = 1, 2$, are called isomorphic if and only if there is an 1-1 and onto map $f : V_1 \to V_2$ such that if $(v, u) \in E_1$ then $(f(u), f(v) \in E_2$.

Example (You can not hear the shape of a graph). There are isospectral non-isomorphic graphs.



The first pair has spectrum $\{\pm 2, 0^3\}$, and the spectrum of the second pair is $\{\pm 1^2, 0, \pm 2\}$.

More sophisticated application

Let G be a k-regular connected graph and $M = (1/d)A$ the normalized adjacency matrix. Notice that M has maximal eigenvalue equal to 1. Let λ_2 be the next eigenvalue. We define the edge expansion of G to be

$$h(G) = \min_{S:|S|\leq|V|/2} \frac{|E(S, V \setminus S)|}{k|S|}$$

where $E(S, V \setminus S) = \partial S$ is the set of edges from the vertices of S to the vertices of $V \setminus S$. To explain edge expansion, let us see two examples.

- If G is not connected, we choose one connected component as S so that $|E(S, V \setminus S)| = 0$. Therefore $h(G) = 0$.

- If G is the complete graph K_n, then $|E(S, V \setminus S)| = |S|.(n - |S|)$ and

$$h(K_n) = \begin{cases} \dfrac{n+1}{2(n-1)}, & n \text{ odd} \\ \dfrac{n}{2(n-1)}, & n \text{ even} \end{cases}$$

  and notice that

$$\lim_{n \to \infty} h(K_n) = \frac{1}{2}.$$

- If G is the circle graph C_n. Let $S = \{1, \ldots, \lfloor n/2 \rfloor\}$. Then $E(S, V \setminus S) = \{((\lfloor n/2 \rfloor, \lfloor n/2 \rfloor \ + 1), (n, 1)\}$. Thus $E(S, V \setminus S) = 2$. Calculating

$$h(C_n) = \begin{cases} \dfrac{2}{n-1}, & n \text{ odd} \\ \dfrac{2}{n}, & n \text{ even} \end{cases}$$

  and notice that

$$\lim_{n \to \infty} h(C_n) = 0.$$

Theorem (Cheeger's Inequality)

For any k-regular graph

$$\frac{1-\lambda_2}{2} \leq h \leq \sqrt{2(1-\lambda_2)}.$$

Thus, we need information on the second eigenvalue.

Theorem (Alon-Boppana)

For every k-regular graph on n-vertices,

$$\lambda_2 \geq 2\sqrt{k-1} - o_n(1)$$

where o_n goes to 0 as $n \to \infty$.

Definition

A sequence of k-regular graphs $(G_i)_{i\in\mathbb{N}}$ of size increasing with i is a family of expanders if there is a constant $\varepsilon > 0$ such that $h(G_i) \geq \varepsilon$ for all i. Notice that any expander graph is a connected graph.

Remark

1. It is a spectral property from Cheeger's Inequality.
2. Expanders have strong connectivity properties.
3. A graph is call Ramanujan if $\lambda_2 \leq 2\sqrt{k-1}$. Ramanujan graphs have the best expansion properties.

Infinite Graphs

We start with the definition of the adjacency matrix analogue for infinite graphs. To any graph we associate vector spaces over $\mathbb{C}$. A natural choice is the vector space with a basis that is the set of vertices $V(G)$. It turns out that the dual of this space is more appropriate in the calculations. We write:

$$C^0(G) = \{f : V(G) \to \mathbb{C} : \ f \text{ a function}\}.$$

Also, similarly to the classical case, we associate to G a Hilbert space:

$$L^2(G) = \left\{ f \in C^0(G) : \sum_{v \in V(G)} |f(v)|^2 < \infty \right\}$$

In the case that G is a finite graph the two spaces coincide. The space $L^2(G)$ admits an inner product:

$$\langle f, f' \rangle = \sum_{v \in V(G)} f(v)\overline{f'(v)} \in \mathbb{C}, \ f, f' \in L^2(G),$$

where the bar denotes the complex conjugate. The inner product induces a norm on $L^2(G)$:

Proposition

Let G be a weighted graph with degrees of vertices uniformly bounded i.e., there is $K > 0$ such that $d_v \leq K$ for all $v \in V(G)$. Then $A(G)$ is a linear operator on $L^2(G)$.

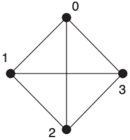
We need to show that, for $f \in L^2(G)$, $A(G)(f) \in L^2(G)$. We check

$$\begin{aligned}\sum_{v\in V(G)} |A(G)(f)(v)|^2 &= \sum_{v\in V(G)} \left(\sum_{u\in V(G)} w(v,u)^2 |f(u)|^2\right) \\ &= \sum_{u\in V(G)} \left(\sum_{v\in V(G)} w(v,u)^2 |f(u)|^2\right)\end{aligned}$$

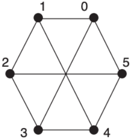
$$
\begin{aligned}
&= \sum_{u\in V(G)} \left(\sum_{v\in V(G)} w(v,u)^2 |f(u)|^2 \right) \\
&= \sum_{u\in V(G)} \left(\sum_{v\in V(G)} w(v,u)^2 \right) |f(u)|^2 \\
&\leq \sum_{u\in V(G)} K^2 |f(u)|^2 \\
&\leq K^2 \sum_{u\in V(G)} |f(u)|^2 < \infty
\end{aligned}
$$

Cayley Graphs

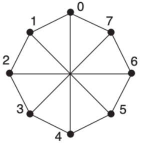
Let G be a group and S a set of generators. We assume that S is symmetric $S = S^{-1}$. The generators of order 2 appear only once. We define a graph $Cay(G, S)$ with set of vertices G and two vertices g and g' are connected by an edge if $g = g's$, $s \in S$. We consider only simple unoriented Cayley graphs.



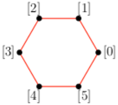
Cay($\mathbb{Z}_4$,{1,2,3})



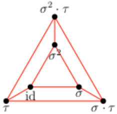
Cay($\mathbb{Z}_6$,{1,3,5})



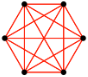
Cay($\mathbb{Z}_8$,{1,4,7})



$\mathrm{Cay}(\mathbb{Z}/6\mathbb{Z}, \{[1]\})$

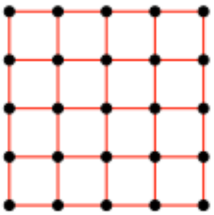


$\mathrm{Cay}(S_3, \{\tau, \sigma\})$



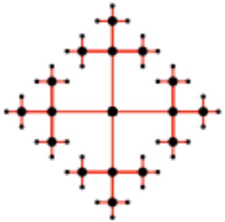
$\mathrm{Cay}(S_3, S_3)$
$\cong \mathrm{Cay}(\mathbb{Z}/6\mathbb{Z}, \mathbb{Z}/6\mathbb{Z})$

and for infinite graphs



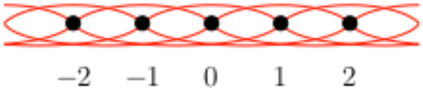
$\mathbb{Z} \times \mathbb{Z}$

$\mathbb{Z}$



$\mathbb{Z} * \mathbb{Z}$

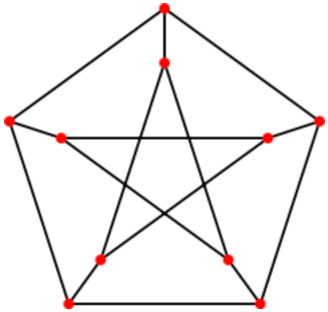
$\mathrm{Cay}(\mathbb{Z}, \{1\})$



$\mathrm{Cay}(\mathbb{Z}, \{2, 3\})$

1. The Cayley graph depends on the generating set. Thus it is not an algebraic invariant.
2. Let $p \geq 23$ be a prime number. Then for each integer number d, $6 \leq d \leq 2p - 7$, there exist at least two non-isomorphic cospectral d-regular Cayley graphs over the dihedral group D_{2p}.
3. A relation in G is represented by a closed path in $Cay(G, S)$.
4. Combinatorial properties of the Cayley graph represent properties of G at infinite (large scale).

Cayley graphs are $|S|$-regular and transitive (for any two vertices there is a graph isomorphism that maps one to the other). But not all regular transitive graphs are transitive. The classical example is the Petersen graph, which is 3-regular, transitive but it is not a Cayley graph.



Since it regular, one of the generators must be of order 2. If it were a Cayley graph the underlying group would be of order 10. There are two groups of order 10: The cyclic group C_{10} and the dihedral group D_{10}, the group of symmetries of the regular pentagon.

$$D_{10} = \{1, r, r^2, r^3, r^4, s, sr, sr^2, sr^3, sr^4\} = \langle r, s : \ r^5 = s^2 = 1, \ srs = r^{-1} \rangle$$

r is the rotation by 72^o and s is a reflection.

- If the group were abelian, then $ab = ba \Rightarrow aba^{-1}b^{-1} = 1$ and that means there are loops of length 4, something that does happen. Thus the group can not be abelian and the only possibility is that $G = D_{10}$.
- Let a be the generator of order 2 and b the generator of order 5. $(ab)^2 = 1$ and again there must be a loop of length 4.
- If all generators have order 2, then no product of 5 generators can be the identity.

Kesten's Theorem

One of the classic theorem in the area is Kesten's Theorem.

Theorem (Kesten, Grigotchuk)

Let G be a simple k-regular graph and A its adjacency operator. Then

$$2\sqrt{k-1} \leq \rho(G) \leq k$$

Also,

1. *G is the regular k-tree if and only if $\rho(G) = 2\sqrt{k-1}$.*
2. *$\rho(G) = k$ if and only if G is amenable.*

1. A graph G is called amenable if it satisfies one of the following equivalent conditions:
    1. (Følner Condition)
    $$\inf_{F, |F| < \infty} \frac{|\partial F|}{|F|} = 0.$$
    2. There is a sequence $\{F_i\}_{i=0}^{\infty}$ of finite subsets so that
    $$\bigcup_{i=0}^{\infty} F_i = V(G)$$
    such that
    $$\lim_{i \to \infty} \frac{|\partial F_i|}{|F_i|} = 0, \text{ or equivalently } \lim_{i \to \infty} \frac{|\partial F_i \cup F_i|}{|F_1|} = 1.$$

1. A group is called amenable if it admits a left-invariant, finitely additive probability measure:
    1. The measure of G is 1.
    2. The measure of a disjoint finite union of subsets is the sum of the measures.
    3. For any subset A of G, the measure of A is equal to the measure of gA, $g \in G$.

    or, equivalently it satisfies the Følner Condition: there are finite subsets of G S_n such that

    $$\lim_{n \to \infty} \frac{|gS_n \triangle S_n|}{|S_n|} = 0, \text{ for each } g \in G.$$

    $(A \triangle B = (A \setminus B) \cup (B \setminus A)$ is the symmetric difference).
2. A group is amenable if and only any Cayley graph is amenable.

Remark

1. Finite graphs are amenable.
2. Infinite trees are not amenable.
3. Abelian groups are amenable.

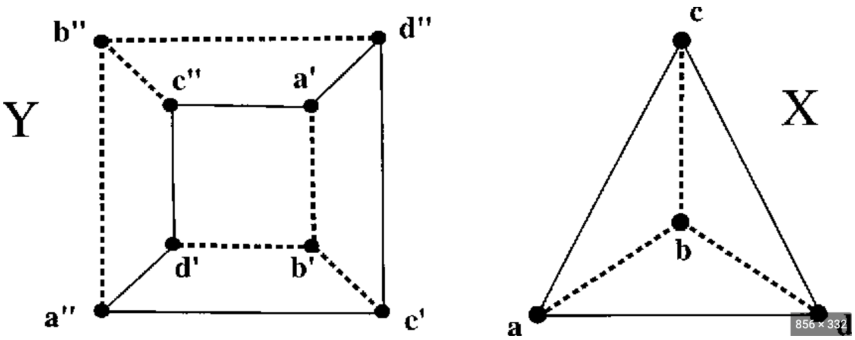
General Question. (Gromov) What kind of information about the group we can derive from the spectrum of its Cayley graph?

Covers of Graphs

A map between graphs $p : G \to H$ is called a graph cover if it is onto, preserves adjacency and, for each $x \in p^{-1}(v)$,

$$p| : N(x) \to N(v)$$

is a bijection.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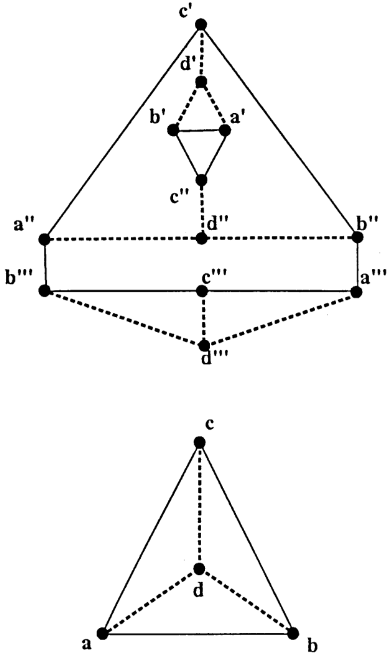

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If G covers H then the characteristic polynomial of H divides the characteristic polynomial of G.
Notice that

$$(A_H)_{p(x),x} = \sum_{y \in p^{-1}(x)} (A_G)xy$$

If we consider the matrices as linear transformations, that means $A_H \circ p = p \circ A_G$.
If u is an eigenvector of H with eigenvalue λ, define $(u \circ p)_y = u_{f(y)}$, an eigenvector of G with the same eigenvalue.

More precisely: Let $V(H) = \{v_1, \dots v_n\}$.
Create the vector $z =^t ({}^t u \, {}^t u \dots {}^t u)$
Then $A_G z = \lambda z$.
In block form,

$$A_G = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mm} \end{pmatrix}$$

Here each row has the same number of 1's as in A_H and they are in the same location except for which block they lie in.
Also

$$\sum_{j=1}^{m} A_{ij} = A_H, \text{ for all } i$$

Thus

$$A_G = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & \dots & A_{mm} \end{pmatrix} \begin{pmatrix} w \\ \vdots \\ w \end{pmatrix} = \begin{pmatrix} (A_{11} + \cdots + A_{1m})w \\ \vdots \\ A_{m1} + \cdots + A_{mm})w \end{pmatrix} = \lambda \begin{pmatrix} w \\ \vdots \\ w \end{pmatrix}$$

Universal Cover

Let G be a graph and v_0 a base vertex. Define a graph T with vertices paths $\sigma = (v_0, v_1, \dots v_n)$ without backtracing. Two paths $\sigma = (v_0, v_1, \dots v_n)$ and $\sigma' = (v_0, v_1', \dots v_m')$ are adjacent if v_n is adjacent to v_m'. Define a map $p : T \to G$ by $p(\sigma) = v_n$.
Then

- T is a tree.
- p is a covering space.

T is called the universal cover of G

Heat Kernel

Definition

The heat kernel on a graph is defined as the solution to the equation

$$\frac{\partial H_t}{\partial t} = -\mathcal{L} H_t.$$

Formally, the heat kernel is given as the operator on $L^2(G)$ by:

$$H_t = e^{-t\mathcal{L}} = \sum_{s=0}^{\infty} (-1)^s \frac{t^s \mathcal{L}^s}{s!}.$$

Remark

If A is diagonalizable, then $A = PDP^{-1}$ with D diagonal. Then

$$e^{-tA} = Pe^{-tD}P^{-1}$$

and $e^{-tD} = \mathrm{diag}(e^{-t\lambda_1}, \dots e^{-t\lambda_n})$. Also, if p_i is the projection to the λ_i-eigenspace,

$$e^{-tA} = \sum_{i=1}^{n} e^{-t\lambda_i} p_i$$

If A is orthogonally diagonalizable, then

$$e^{-tA}(x, y) = \sum_{i=1}^{n} e^{-t\lambda_i} \phi_i(x)\phi_i(y)$$

for the orthogonal eigenfunctions ϕ_i.

If $p : G \to H$ is a covering map, $\pi(x) = u$ then

$$h_t^H(x, y) = \sum_{v \in p^{-1}y} h_t^G(u, v)$$

Let $\mathcal{G}$ be a k-regular graph. Let $\mathcal{L}$ denote the normalised Laplacian on $\mathcal{G}$. We set $H_t = e^{-t\mathcal{L}}$. Remember that a *walk of length* s is a sequence of vertices of $\mathcal{G}$

$$W_s = (v_0, \dots, v_s), \; v_j = v_{j+1} \text{ or } \{v_j, v_{j+1}\} \text{ is an edge}$$

Let x and y be vertices. Let $\rho_{x,y}(s, r)$ be the number of walks of length s such that $v_0 = x$, $v_s = y$ that contain exactly r edges.

Proposition

With the above notation:

$$H_t(x,y) = \sum_{s=0}^{\infty} (-1)^s \frac{t^s}{s!} \left(\sum_r (-1)^r \left(\frac{1}{k}\right)^r \rho_{x,y}(s,r) \right)$$

Notice that

$$H_t = I - t\mathcal{L} + \frac{t^2\mathcal{L}^2}{2!} - \dots$$

In this case,

$$\mathcal{L}^s(x,y) = \left(I - \frac{1}{k}A \right)^s = \sum_{r=0}^{s} \binom{s}{r} (-1)^r \left(\frac{1}{k}\right)^r A^r(x,y)$$

But $A^r(x,y)$ is the number of edge-paths from x to y that contain exactly r edges. But

$$\rho_{x,y}(s,r) = \binom{s}{r} A^r(x,y)$$

because $\rho_{x,y}(s,r)$ is the number of walks containing exactly r edges. The result follows.

Corollary

Let $\mathcal{G}$ be the "integer lattice" (the infinite 2-regular tree). Then the heat kernel satisfies:

$$H_t(x,x) = \sum_{s=0}^{\infty} (-1)^s \binom{2s}{s} \left(\frac{1}{2}\right)^s \frac{t^s}{s!}.$$

Identify the vertex set of $\mathcal{G}$ with the integers $\mathbb{Z}$. Define

$$Y : V(\mathcal{G}) = \mathbb{Z} \to \mathbb{C},\ Y(x) = x + 1.$$

Then,

$$\mathcal{L} = I - Y/2 - Y^{-1}/2.$$

The coefficient of t^s is given by the coefficient of Y^0 in $\mathcal{L}^s$ i.e. by

$$\sum_{r=0}^{[s/2]} \frac{s!}{(s-2r)!r!r!} \frac{1}{2^{2r}} = \sum_{r=0}^{[s/2]} \binom{s}{2r} \binom{2r}{r} \frac{1}{2^{2r}}.$$

To finish the proof, consider the identity:

$$\left(x + \frac{1}{2}\right)^{2s} = \left(\left(x + \frac{1}{2}\right)^2\right)^s = \left(x^2 + x + \frac{1}{4}\right)^s.$$

We look at the coefficient of x^s in both sides: Left Side:

$$\left(\frac{1}{2}\right)^s \binom{2s}{s}$$

Right Side:

$$\sum_{i+j+k=s} \frac{s!}{i!j!k!} \frac{1}{4^k}$$

where $2i + j = s$. That implies $i = k$ and $j = s - 2k$. Rewriting the coefficient:

$$\sum_{k} \frac{s!}{k!(s-2k)!k!} \frac{1}{4^k} = \sum_{k=0}^{[s/2]} \frac{s!}{(s-2k)!k!k!} \frac{1}{2^{2k}}.$$

The result follows.

Different interpretations:

- It is related to *hypergeometric functions*:

$$H_t(x,x) = {}_1F_1(1/2;1;-2t) = \sum_{s=0}^{\infty} \frac{\frac{1}{2}\frac{3}{2}\dots\frac{2s-1}{2}}{(s!)^2}(-2t)^s.$$

To see this notice that

$$\sum_{s=0}^{\infty} \frac{\frac{1}{2}\frac{3}{2}\dots\frac{2s-1}{2}}{(s!)^2}(-2t)^s \quad = \quad \sum_{s=0}^{\infty} \frac{1.3.\dots(2s-1)}{2^s(s!)^2}(-2t)^s$$

$$
\begin{aligned}
&= \sum_{s=0}^{\infty}(-1)^s \frac{(2s)!}{(2.4.\ldots.(2s))(s!)^2} t^s \\
&= \sum_{s=0}^{\infty}(-1)^s \frac{(2s)!}{2^s(s!)(s!)^2} t^s \\
&= \sum_{s=0}^{\infty}(-1)^s \frac{(2s)!}{(s!)(s!)} \left(\frac{1}{2}\right)^s \frac{t^s}{s!} \\
&= \sum_{s=0}^{\infty}(-1)^s \binom{2s}{s} \left(\frac{1}{2}\right)^s \frac{t^s}{s!}
\end{aligned}
$$

- It is connected to *Bessel functions*:

$$H_t(x,x) = e^{-t} I_0(-t) = e^{-t} \sum_{r=0}^{\infty} \frac{t^{2r}}{(r!)^2 2^{2r}}.$$

We use the classical expansion of e^{-t} and multiply the two series. The coefficient of t^s will be

$$\begin{aligned}
\sum_{k+2r=s} (-1)^k \frac{1}{k!(r!)^2 2^{2r}} &= \frac{1}{s!} \sum_{r=0}^{[s/2]} (-1)^{s-2r} \frac{s!}{(s-2r)!(r!)^2 2^{2r}} \\
&= \frac{1}{s!} \sum_{r=0}^{[s/2]} (-1)^{s} \frac{s!}{(s-2r)!(r!)^2 2^{2r}}
\end{aligned}$$

Lattice Graph

The eigenvalues of the n-cycle are

$$\lambda_k = 2\sin^2\left(\frac{\pi k}{n}\right),\ k = 0, 1, \ldots, n-1.$$

Also, the heat kernel of the infinite line is the limit of the heat kernel of the n-cycle as $n \to \infty$. Therefore

$$\begin{aligned} H_t(x,x) &= \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} e^{-t\lambda_k} = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \exp\left(-2t\sin^2\left(\frac{\pi k}{n}\right)\right) \\ &= \frac{1}{\pi}\int_0^{\pi} e^{-2t\sin^2 x} dx = \frac{2}{\pi}\int_0^{\pi/2} e^{-2t\sin^2 x} dx \\ &= \frac{\sqrt{2}}{\pi\sqrt{t}}\int_0^{\sqrt{2t}} \frac{e^{-y^2} dy}{\sqrt{1-\frac{y^2}{2t}}} \end{aligned}$$

For the heat kernel of the path, look at the coefficient of Y^a in $\mathcal{L}^k$ above:

$$H_t(x, x+a) = H_t(x, x-a) = (-1)^a \sum_{k=a}^{\infty} (-1)^k \binom{2k}{k+a} \frac{t^k}{2^k k!}.$$

Differently,

$$H_t(x, x+a) = \left(\frac{t}{2}\right)^a {}_1F_1(1/2+a, 1+2a, -2t) = (-1)^a e^t I_a(-t).$$

Thus the heat kernel satisfies

$$H_t(x, x+a) = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{-t\sin^2 x} \cos(2ax) dx.$$

Let $H^{(n)}$ be the heat kernel on the n-cycle. Then

$$H_t^{(n)}(x,y) = \sum_{k=-\infty}^{\infty} H_t(x, y+kn).$$

That follows from the fact that walks on C_n from x to y can be identified with walks from x to $y+nk$, for some $k \in \mathbb{Z}$. Combining all the above formulas:

$$\sum_{k=0}^{\infty} \sum_{j=-\infty}^{\infty} (-1)^{nj+k} \binom{2k}{k+nj} \frac{t^k}{2^k k!} = \frac{1}{n} \sum_{k=0}^{n-1} e^{-2t \sin^2 \frac{\pi k}{n}}.$$

Remark (Chung-Yau)

1. The heat kernel of the k-regular tree is given by:

$$H_t(0,0) = \frac{2k(k-1)}{\pi} \int_0^{\pi} \frac{e^{-t(1-\frac{2\sqrt{k-1}}{k}\cos x)} \sin^2 x}{k^2 - 4(k-1)\cos^2 x} dx$$

$$H_t(0,a) = \\ \frac{2k(k-1)}{\pi} \int_0^{\pi} \frac{e^{-t(1-\frac{2\sqrt{k-1}}{k}\cos x)} \sin x[(k-1)\sin(a+1)x - \sin(a-1)x]}{k^2 - 4(k-1)\cos^2 x} dx$$

2. For a k regular graph G, its heat kernel satisfies

$$h_t(u,v) = \sum_{y \in \pi^{-1}(u)} H_t(x,y)$$

where $\pi : G \to T_k$ is the universal cover of G, $\pi(x) = u$, and H_t is the heat kernel of the k-regular tree.

Combinatorial Covers

Let $\tilde{G}$ and G be two weighted directed graphs. We say $\tilde{G}$ is a combinatorial covering of G (or G is covered by $\tilde{G}$) if there is a mapping $\pi : V(\tilde{G}) \to V(G)$ satisfying the following two properties:

(i) There is an $n \in \mathcal{R}$, called the index of π, such that for $u, v \in V(G)$, we have

$$\sum_{x \in \pi^{-1}(u)} \sum_{y \in \pi^{-1}(v)} w(x, y) = nw(u, v), \sum_{x \in \pi^{-1}(u)} \sum_{y \in \pi^{-1}(v)} w(y, x) = nw(v, u$$

(ii) For $x, y \in V(\tilde{G})$ with $\pi(x) = \pi(y)$ and $v \in V(G)$, we have

$$\sum_{z \in \pi^{-1}(v)} w(z, x) = \sum_{z' \in \pi^{-1}(v)} w(z', y), \sum_{z \in \pi^{-1}(v)} w(x, z) = \sum_{z' \in \pi^{-1}(v)} w(y, z')$$

Remark

- The map π is called also equitable partition.
- A graph $\tilde{G}$ is said to be a regular covering of G if for a fixed vertex $v \in V(G)$ and for any vertex x of $V(\tilde{G})$, $\tilde{G}$ is a covering of G under a mapping π_x which maps x to v.In addition, if $\pi^{-1}(v) = \{x\}$, we say that $\tilde{G}$ is a strong regular covering of G.
- That implies
  1. For $x \in \pi^{-1}(v)$,
  $$|\pi^{-1}(v)| \sum_{z \in \pi^{-1}(u)} w(z,x) = mw(u,v)$$
  2. For $x, y \in \pi^{-1}(v)$, $d_x = d_y$

For a function $f : V(G) \to \mathbb{C}$, we have

$$L(f)(v) = \sum_u (f(v) - f(u))w(u,v)$$

and $\mathcal{L} = T^{-1/2} L T^{-1/2}$, where T is the diagonal matrix with v entry d_v.
(1) If $\tilde{G}$ is a cover of G, then an eigenvalue of G is an eigenvalue of $\tilde{G}$ (for $\mathcal{L}$):
Let λ be an eigenvalue of G, with eigenfunction g. Set $f = T^{-1/2}g$, the harmonic eigenfunction. Then

$$\sum_u (f(v) - f(u))w(u,v) = T^{-1/2} Lg(v) = T^{-1/2} L T^{-1/2} T^{1/2} g(v) = \mathcal{L} T^{1/2} g(v) = \lambda T^{1/2} g(v) = \lambda T f(v) = \lambda d_v f(v).$$

We cal lift f to $\tilde{G}$ by defining $\tilde{f}(x) = f\pi(x)$. Then

$$\sum_y (\tilde{f}(x) - \tilde{f}(y))w(x,y) = \sum_v (f(u) - f(v))w(u,v) = \lambda d_x \tilde{f}(x).$$

(2) If a harmonic eigenfunction f of $\tilde{G}$ with eigenvalue λ has a non-trivial image in G, then λ is an eigenvalue of G.
For a harmonic eigenfuntion f with eigenvalue λ on $\tilde{G}$, and $x \in \pi^{-1}(v)$:

$$\sum_{y}(f(x) - f(y))w(x, y) = \lambda f(x)d_x$$

By summing over all $x \in \pi^{-1}(v)$:

$$\sum_{x\in\pi^{-1}(v)} \sum_{y}(f(x) - f(y))w(x, y) = \lambda \sum_{x\in\pi^{-1}(v)} f(x)d_x$$

We define the image of f as πf

$$\pi f(v) = \sum_{x\in\pi^{-1}(v)} \frac{f(x)d_x}{d_v}$$

Then for $g = \pi f$ we have

$$\sum_{u}(g(v) - g(u))w(u, v) = \lambda g(v)d_v$$

So: if $\tilde{G}$ is a strongly regular combinatorial cover of G, the two graphs have the same spectrum for the combinatorial Laplacian.

Let T_k denote the k-regular tree. Let P be the infinite weighted path: $V(P) = \{0, 1, 2, \dots\}$ and $w(j, j-1) = k(k-1)^{j-1}$.
For the Laplacian of P,

$$\mathcal{L} = \begin{cases} 1, & \text{if } i = j \\ -\dfrac{1}{\sqrt{k}} & \text{if } (i,j) = (1,0) \text{ or } (i,j) = (0,1) \\ -\dfrac{k-1}{\sqrt{k}} & \text{if } |i-j| = 1, i,j \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

To compute the spectrum of P, we crop P at ℓ and then take the limit as $\ell \to \infty$. So

$$V(P^{\ell}) = \{0, 1, \ldots, \ell\}$$

For the eigenvalues of the Laplacian of P^{ℓ} calculate:

$$\lambda I - \mathcal{L}_{\mathcal{P}(\ell)} = \begin{vmatrix} \lambda - 1 & \frac{1}{\sqrt{k}} & 0 & \ldots & \ldots & 0 \\ \frac{1}{\sqrt{k}} & \lambda - 1 & \frac{\sqrt{k-1}}{k} & \ldots & \ldots & \ldots \\ 0 & \frac{\sqrt{k-1}}{k} & \lambda - 1 & \ddots & \ldots & \ldots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ldots \\ \vdots & \vdots & \vdots & \ddots & \lambda - 1 & \sqrt{\frac{k-1}{k}} \\ 0 & \vdots & \vdots & \vdots & \sqrt{\frac{k-1}{k}} & \lambda - 1 \end{vmatrix}$$

Set:

$$\boxed{\alpha = \frac{\sqrt{k-1}}{k}, \;\; x = \frac{\lambda - 1}{2\alpha}}$$

Then, direct calculation shows that:

$$\boxed{\det(\lambda I - \mathcal{L}_{\mathcal{P}(\ell)}) = \alpha^{\ell-1}(4x^2\alpha^2 - 1)U_{\ell-1}(x)}$$

were $U_{\ell-1}(x)$ is the **Chebyshev Polynomial of Second Type**:
Now that we have a closed form of the determinant we find that the eigenvalues are:

$$\boxed{0, 2, 1 - 2\alpha\cos\left(\frac{\pi j}{\ell}\right), \; j = 1, 2, \ldots, \ell - 1}$$

That suggests that as $\ell \to \infty$, the eigenvalues of $\mathcal{L}_{\mathcal{P}}$ are

$$\operatorname{spec}(\mathcal{L}_{\mathcal{P}}) = \left[1 - \frac{2\sqrt{k-1}}{k}, 1 + \frac{2\sqrt{k-1}}{k}\right]$$

Write the eigenvalues in the form:

$$\lambda = 1 - 2\alpha\rho, \quad \rho \in \mathbb{R}.$$

For $\lambda = 0$, the eigenfuncion is $\phi_0 = f_0/\|f_0\|$ where

$$f_0(p) = \begin{cases} 0, & \text{for } p = 0 \\ \sqrt{k(k-1)^{p-1}}, & \text{for } 1 \leq p \leq \ell \\ \sqrt{(k-1)^{\ell-1}}, & \text{for } p = \ell \end{cases}$$

For $\lambda = 2$, the eigenfuncion is $\phi_\ell = f_\ell/\|f_\ell\|$ where

$$f_\ell(p) = \begin{cases} 1, & \text{for } p = 0 \\ (-1)^p\sqrt{k(k-1)^{p-1}}, & \text{for } 1 \leq p \leq \ell \\ (-1)^\ell\sqrt{(k-1)^{\ell-1}}, & \text{for } p = \ell \end{cases}$$

The eigenfunction ϕ_j, for $j = 1, \dots, \ell - 1$ associated with $1 - \frac{2\sqrt{k-1}}{k} \cos \frac{\pi j}{\ell}$ is $f_j / \|f_j\|$ where

$$f_j(p) = \begin{cases} \sqrt{\dfrac{k}{k-1}} \sin \dfrac{\pi j}{\ell}, & \text{for } p = 0 \\ \sin \dfrac{\pi j(p+1)}{\ell} - \dfrac{1}{k-1} \sin \dfrac{\pi j(p-1)}{\ell}, & \text{for } 1 \leq p \leq \ell \\ -\dfrac{\sqrt{k}}{k-1} \sin \dfrac{\pi j}{\ell} & \text{for } p = \ell \end{cases}$$

Also,

$$\|f_j\|^2 = \frac{\ell k^2}{2(k-1)^2} \left(1 - \frac{4(k-1)}{k^2} \cos^2 \frac{\pi j}{\ell} \right)$$

And the heat kernel is

$$h^{(\ell)}(0,0) = \sum_{j=1}^{\ell-1} \frac{e^{-t(1-\frac{2\sqrt{k-1}}{k} cos\frac{\pi j}{\ell})} \sin^2 \frac{\pi j}{\ell}}{\frac{\ell k}{2(k-1)} \left(1 - \frac{4(k-1)}{k^2} \cos^2 \frac{\pi j}{\ell}\right)} + \frac{1}{\|f_0\|^2} + \frac{1}{\|f_1\|^2}$$

Homesick Random Walks

Starting from a simple graph, Lyons defined a weight function on a graph G that determines a homesick random walk on G. He defines weights as

$$w(u,v) := \begin{cases} \dfrac{\mu}{d_u + (\mu - 1)d_u^-}, & v \in S_{r-1}(z) \\ \dfrac{1}{d_u + (\mu - 1)d_u^-}, & \text{otherwise} \end{cases}$$

where μ is the homesick parameter. Here, the degree and the sphere are the regular combinatorial objects. With this weight, G becomes a weighted directed graph.

The basic result is

Theorem (Lyons)

For $G = Cay(\Gamma)$, if $\mu < gr(\Gamma)$, then RW_μ is transient. If $\mu > gr(\Gamma)$, then RW_μ is positive recurrent.

1. $gr(G)$ refers to the *growth* of G and is defined as:

$$gr(G) = \lim_{n \to \infty} |S_n(e)|^{1/n},$$

where the sphere is the combinatorial sphere centered at the identity e.

2. On a random walk, "transient" means that at some point you will never return to the origin of the walk. "Positive recurrent" means that you will always eventually return to the origin of the walk.

Let $l_a = d(0, a)$ and $l_b = d(0, b)$, where 0 denotes the root of the tree. Let S be the set of branch roots for branches which contain both a and b. Let $D = \max\{d(0, s) | s \in S\}$. $D = 0$ if S is the empty set. Then the heat kernel of the k-regular tree is given by:

I: *If* $\mu > k-1$ *:*

i:

$$h_t(0,0) = \frac{(\mu-k+1)(1+e^{-2t})}{2\mu} + \frac{2(k-1)}{\pi}\int_0^{\pi} \frac{e^{-t[1-\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\sin^2 x}{(\mu+k-1)[1-\frac{4\mu(k-1)}{(\mu+k-1)^2}\cos^2 x]}\,dx$$

ii:

$$h_t(0,a) = \frac{(\mu-k+1)\sqrt{\frac{\mu+k-1}{\mu^{l_a}}}\left[1+(-1)^{l_a}\cdot e^{-2t}\right]}{2\mu\sqrt{k}}$$

$$+\frac{2(k-1)^2}{\pi\sqrt{k\cdot(k-1)^{l_a}}}\int_0^{\pi}\frac{e^{-t[1-\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\left(\sqrt{\mu+k-1}\sin x\right)\left(\sin[(l_a+1)x]-\frac{\mu}{k-1}\sin[(l_a-1)x]\right)}{(\mu+k-1)^2[1-\frac{4\mu(k-1)}{(\mu+k-1)^2}\cos^2 x]}\,dx$$

iii: *if* $D=0$

$$h_t(a,b) = -\frac{1}{k}\eta^{\infty}(l_a-1,l_b-1) + \frac{(\mu-k+1)(\mu+k-1)(\sqrt{\mu})^{-(l_a+l_b)}\left[1+(-1)^{(l_a+l_b)}\cdot e^{-2t}\right]}{2\mu k}$$

$$+\int_0^{\pi}\frac{e^{-t[1-\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\left(\sin[(l_a+1)x]-\frac{\mu}{k-1}\sin[(l_a-1)x]\right)\left(\sin[(l_b+1)x]-\frac{\mu}{k-1}\sin[(l_b-1)x]\right)}{\frac{k\pi}{2(k-1)^2}\sqrt{(k-1)^{l_a+l_b-2}}(\mu+k-1)^2[1-\frac{4\mu(k-1)}{(\mu+k-1)^2}\cos^2 x]}\,dx$$

iv: *if* $D > 0$ *and* $l_a, l_b \neq D$

$$h_t(a,b) = \frac{k-1}{k}\eta^\infty(l_a-1,l_b-1) - \frac{1}{k-1}\eta^\infty(l_a-D-1,l_b-D-1) + \sum_{y=1}^{D}\frac{k-2}{k-1}\eta^\infty(l_a-y,l_b-y)$$

$$+\frac{(\mu-k+1)(\mu+k-1)(\sqrt{\mu})^{-(l_a+l_b)}\left[1+(-1)^{(l_a+l_b)}\cdot e^{-2t}\right]}{2\mu k}$$

$$+\int_0^\pi \frac{e^{-t[1-\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\left(\sin[(l_a+1)x]-\frac{\mu}{k-1}\sin[(l_a-1)x]\right)\left(\sin[(l_b+1)x]-\frac{\mu}{k-1}\sin[(l_b-1)x]\right)}{\frac{k\pi}{2(k-1)^2}\sqrt{(k-1)^{l_a+l_b-2}}(\mu+k-1)^2[1-\frac{4\mu(k-1)}{(\mu+k-1)^2}\cos^2 x]}\,dx$$

v: *if* $D > 0$ *and* $l_a = D$ *or* $l_b = D$

$$h_t(a,b) = \frac{k-1}{k}\eta^\infty(l_a-1,l_b-1) + \sum_{y=1}^{D}\frac{k-2}{k-1}\eta^\infty(l_a-y,l_b-y)$$

$$+\frac{(\mu-k+1)(\mu+k-1)(\sqrt{\mu})^{-(l_a+l_b)}\left[1+(-1)^{(l_a+l_b)}\cdot e^{-2t}\right]}{2\mu k}$$

$$+\int_0^\pi \frac{e^{-t[1-\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\left(\sin[(l_a+1)x]-\frac{\mu}{k-1}\sin[(l_a-1)x]\right)\left(\sin[(l_b+1)x]-\frac{\mu}{k-1}\sin[(l_b-1)x]\right)}{\frac{k\pi}{2(k-1)^2}\sqrt{(k-1)^{l_a+l_b-2}}(\mu+k-1)^2[1-\frac{4\mu(k-1)}{(\mu+k-1)^2}\cos^2 x]}\,dx$$

II: *If* $\mu \leq k-1$:

i:

$$h_t(0,0) = \frac{2(k-1)}{\pi}\int_0^\pi \frac{e^{-t[1-\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\sin^2 x}{(\mu+k-1)[1-\frac{4\mu(k-1)}{(\mu+k-1)^2}\cos^2 x]}\,dx$$

ii: $h_t(0,a) =$

$$\frac{2(k-1)^2}{\pi\sqrt{k\cdot(k-1)^{l_a}}}\int_0^\pi \frac{e^{-t[1-\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\left(\sqrt{\mu+k-1}\sin x\right)\left(\sin[(l_a+1)x]-\frac{\mu}{k-1}\sin[(l_a-1)x]\right)}{(\mu+k-1)^2[1-\frac{4\mu(k-1)}{(\mu+k-1)^2}\cos^2 x]}\,dx$$

iii: *if* $D = 0$

$$h_t(a,b) = -\frac{1}{k}\eta^{\infty}(l_a - 1, l_b - 1)$$

$$+\int_0^{\pi} \frac{e^{-t[1-\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\left(\sin[(l_a+1)x] - \frac{\mu}{k-1}\sin[(l_a-1)x]\right)\left(\sin[(l_b+1)x] - \frac{\mu}{k-1}\sin[(l_b-1)x]\right)}{\frac{k\pi}{2(k-1)^2}\sqrt{(k-1)^{l_a+l_b-2}}(\mu+k-1)^2[1-\frac{4\mu(k-1)}{(\mu+k-1)^2}\cos^2 x]}\,dx$$

iv: *if* $D > 0$ *and* $l_a, l_b \neq D$

$$h_t(a,b) = \frac{k-1}{k}\eta^{\infty}(l_a - 1, l_b - 1) - \frac{1}{k-1}\eta^{\infty}(l_a - D - 1, l_b - D - 1) + \sum_{y=1}^{D}\frac{k-2}{k-1}\eta^{\infty}(l_a - y, l_b - y)$$

$$+\int_0^{\pi} \frac{e^{-t[1-\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\left(\sin[(l_a+1)x] - \frac{\mu}{k-1}\sin[(l_a-1)x]\right)\left(\sin[(l_b+1)x] - \frac{\mu}{k-1}\sin[(l_b-1)x]\right)}{\frac{k\pi}{2(k-1)^2}\sqrt{(k-1)^{l_a+l_b-2}}(\mu+k-1)^2[1-\frac{4\mu(k-1)}{(\mu+k-1)^2}\cos^2 x]}\,dx$$

v: *if* $D > 0$ *and* $l_a = D$ *or* $l_b = D$

$$h_t(a,b) = \frac{k-1}{k}\eta^{\infty}(l_a - 1, l_b - 1) + \sum_{y=1}^{D}\frac{k-2}{k-1}\eta^{\infty}(l_a - y, l_b - y)$$

where:

$$\begin{aligned}
\eta^{\infty}(0,0) &= \frac{2}{\pi}\int_0^{\pi} e^{-t[1+\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\sin^2 x\,dx \\
\eta^{\infty}(0,1) &= \frac{4}{\pi\sqrt{k-1}}\int_0^{\pi} e^{-t[1+\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\sin^2 x\cos x\,dx \\
\eta^{\infty}(0,m) &= \frac{2}{\pi(\sqrt{k-1})^{m}}\int_0^{\pi} e^{-t[1+\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\sin\left[(m+1)x\right]\sin x\,dx, \qquad \textit{where } m>1 \\
\eta^{\infty}(1,1) &= \frac{8}{\pi(k-1)}\int_0^{\pi} e^{-t[1+\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\sin^2 x\cos^2 x\,dx \\
\eta^{\infty}(1,m) &= \frac{4}{\pi(\sqrt{k-1})^{m+1}}\int_0^{\pi} e^{-t[1+\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\sin x\sin\left[(m+1)x\right]\cos x\,dx, \qquad \textit{where } m>1 \\
\eta^{\infty}(m,n) &= \frac{2}{\pi(\sqrt{k-1})^{m+n}}\int_0^{\pi} e^{-t[1+\frac{2\sqrt{\mu\cdot(k-1)}}{\mu+k-1}\cos x]}\sin\left[(m+1)x\right]\sin\left[(n+1)x\right]dx, \qquad \textit{where } m,n>1.
\end{aligned}$$

Riemann Zeta Function

- $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod{}_{p,\,\text{prime}} (1 - p^{-s})^{-1}, \quad Res(s) > 1.$$
- It has a simple pole at $s = 1$.
- Riemann Hypotheses. All the non-trivial zeros of $\zeta(s)$ are on the line $Re(s) = 1/2$.

- Prime Number Theorem (Hadamard-Poussin)

$$\#\{p\,\text{prime},\ p \leq x\} \sim \frac{x}{\log x} \text{ as } x \to \infty$$

- For a relation in the zeta function

$$\Lambda(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)(s) = \Lambda(1-s)$$

where

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$$

- There is a relation between primes and the zeros of the zeta function.
- The trivial zeros are the even negative integers.

Let $C = (v_1, \dots v_m)$ be a closed backtrackless tailess path on a graph X

- A path has backtracking if $v_{j-1} = v_{j+1}$ for some $1 \leq j \leq m-1$. If a path does not have bactracking is called proper.
- A tail is a vertex of degree 1.
- $\nu(C) = m - 1$ is the number of edges in the path.
- If C is a circle ($v_m = v_1$), then it called primitive if C can not be of the form D^K, for a circle D.

We define an equivalence relation between two such circles $C = (v_1, \dots v_m = v_1)$, $D = (w_1, \dots w_m = w_1)$ if $w_j = v_{j+k}$ for all j. Let $[C]$ denote the equivalence class.
The Ihara Zeta Function on X is defined as

$$Z_X(u) = \prod_{[C]} (1 - u^{\nu(C)})^{-1}$$

C is a closed backtrackless tailess primiotive path.

The Euler characteristic of X is defined as

$$\chi(X) = |V| - |E| = 1 - r, \; r \text{ the rank of } \pi_1(X)$$

If X is $(q+1)$-regular graph on n vertices, then

$$-\chi(X) = r - 1 = \frac{n(q-1)}{2}$$

Notice that $|E| = n(q+1)/2$

Example. Let $X = C_n$. Then $Z_x(u) = (1 - u^n)^2$

Theorem

$$Z_X(u)^{-1} = (1 - u^2)^{r-1} \det(I + Au + qu^2 I)$$

Set $u = q^{-s}$. Then $Z_X(q^{-s})$ is said to satisfy the Riemann Hypothesis if and only if $Z_X(q^{-s}) = 0$, $Res \in (0, 1)$ implies that $Res = 1/2$.
The graph X is Ramanujan if every eigenvalue λ, $|\lambda| \neq q + 1$, $|\lambda| \leq 2\sqrt{q}$.

Let $\phi(z) = \det(zI - A)$, the characteristic polynomial of A. Notice that

$$\begin{aligned} Z_X(u)^{-1} &= (1-u^2)^{r-1} \det((1+qu^2)I + Au) \\ &= (1-q^{-s})^{r-1} \det((1+q^{1-2s}I - q^{-s}A) \end{aligned}$$

We see that s is a pole, then $\det((q^s + q^{1-s})I - A) = 0$ Thus s is a pole if and only if $\lambda = q^s + q^{1-s}$ is an eigenvalue of A. In particular it must be real because A is symmetric. Also in absolute value is at most $q+1$.
If $Re(s) = a = 1/2$,

$$\lambda = \sqrt{q}(q^{ib} - q^{-ib}) = 2\sqrt{q}\cos(b \log q)$$

If $Re(s) = a \neq 1/2$

$$\lambda^2 = q^{2a} + q^{2-2a} + 2q$$

If it does not satisfy the Riemann hypothesis. Let $s = a + ib$ is a pole, $a \neq 1/2$, $0 < a < 1$. We have an eigenvalue λ with $\lambda^2 = q^{2a} + q^{2-2a} + 2q$. The function $q \to q^{2a} + q^{2-2a} + 2q$ on $[0, 1]$ has a unique minimum at $a = 1/2$, with value $4q$ and its maxima at 0 and 1 with value $(q+1)^2$. So

$$2\sqrt{q} < |\lambda| < q + 1$$

and the graph is not Ramanujan.

Suppose now that the Riemann hypothesis is satisfied. If $\lambda = q^s + q^{1-s}$ is an eigenvalue and $s = a + ib$ a pole. If $a \neq 1/2$ then either $a \leq 0$ or $a \geq 1$ and $\lambda^2 = q^{2a} + q^{2-2a} + 2q$. Thus $|\lambda| \geq q + 1$ and so $|\lambda| = q + 1$.
If $a = 1/2$, then $\lambda = 2\sqrt{q}\cos(b \log q)$, and so $|\lambda| \leq 2\sqrt{q}$.

For a regular finite graph X the Ihara zeta function can be written as

$$Z_X(u) = exp\left(\sum_{r=1}^{\infty} \frac{c_r t^r}{r}\right)$$

where c_r is the number of closed oriented loops of length r in X.
Notice that there d representatives in $[C]$, if the length of $[C]$ is d and

$$\sum_{d|r} \sum_{C,|C|=d} = c_r, \; C \text{ primitive}$$

$$
\begin{aligned}
\ln Z_X(u) &= -\sum_{[C]} \ln(1 - u^{|C|}) = \sum_{[C]} \sum_{j \geq 1} \frac{1}{j} u^{|C|j} = \sum_{j \geq 1} \sum_{d \geq 1} \sum_{[C], |C| = d} \frac{1}{j} u^{dj} \\
&= \sum_{j \geq 1} \sum_{d \geq 1} \sum_{[C], |C| = d} \frac{1}{dj} u^{dj} = \sum_{r \geq 1} \frac{c_r}{r} u^r
\end{aligned}
$$

The distance of based graphs (X_i, v_i), $i = 1.2$ is given by

$$D((X_1, v_1), (X_2, v_2)) = \inf \left\{ \frac{1}{n+1} : B_{X_1}(v_1) \text{ is isometric to } B_{X_2}(v_2) \right\}$$

Let X be the limit of a sequence of k-regular graphs such that there are covering maps $X_n \to X_{n+1}$. Then define

$$\ln Z_X(u) = \lim_{n\to\infty} \frac{1}{|X_n|} \ln_{X_n}(u) = \sum_{r=1}^{\infty} \frac{\tilde{c}_r u^r}{r}, \text{ where } \tilde{c}_r = \lim_{n\to\infty} \frac{c_r(X_n)}{|X_n|}$$

and the radius of convergence is at least $1/k - 1$

With the notation and assumption as before, for $|u| < 1/k-1$, the following holds

$$\ln Z_X(u) = -\frac{k-2}{2}\ln(1-u^2) - \int_{-1}^{1} \ln(1-ku\lambda+(k-1)u^2)d\mu(\lambda)$$

For groups, if P is the uniform random walk on the Cayley graph of the group X

$$\ln Z_X(u) = -\frac{k-2}{2}\ln(1-u^2) - tr\ln(I - tuP + (k-1)u^2 I)$$